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Theoretical Model of Gravitational Perturbation of Current Collector Axisymmetric Flow Field

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Some designs of liquid metal current collectors in homopolar motors and generators are essentially rotating liquid metal fluids in cylindrical channels with free surfaces and will, at critical rotational speeds, become unstable. An investigation ~~at David Taylor Research Center (DTRC)~~ is being performed to understand the role of gravity in modifying this ejection instability. Some gravitational effects can be theoretically treated by perturbation techniques on the axisymmetric base flow of the liquid metal. This leads to a modification of previously calculated critical current collector ejection values neglecting gravity effects. The purpose of this report is to document the derivation of the mathematical model which determines the perturbation of the liquid metal base flow due to gravitational effects. Since gravity is a small force compared with the centrifugal effects, the base flow solutions can be expanded in inverse powers of the Froude number and modified liquid flow profiles can be determined as a function of the azimuthal angle. This model will be used in later work to theoretically study the effects of gravity on the ejection point of the

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current collector. A rederivation of the hydrodynamic instability threshold of a liquid metal current collector is presented in an Appendix to this report.

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CONTENTS

FIGURES

SYMBOLS

F_r = Froude number defined as (V_o^2/gR_o), where $F_r >> 1$. This nondimensional number is the ratio of the inertial force to the gravity force.

G = current collector dimensional radial gap (Fig. 1).

H = current collector dimensional axial gap (Fig. 1).

L = dimensional length along current collector rotor at free surface to tip of rotor (Fig. 1).

r^* = dimensional radial coordinate from axis of rotor where origin of coordinate system is at O' (Fig. 1).

x = $(r^* - R_o)/H$ = nondimensional radial coordinate from free surface when coordinate system is located at point O (Fig. 1).

z = nondimensional axial coordinate where origin of coordinate system is at O' (Fig. 1).

$\dfrac{\partial}{\partial r^*}$ = $\dfrac{1}{H}\dfrac{\partial}{\partial x}$

R_o = dimensional radial coordinate from axis of rotor to point where free surface touches rotor where origin of coordinate system is at O' (Fig. 1).

$\dfrac{\partial}{\partial z^*}$ = $\dfrac{1}{H}\dfrac{\partial}{\partial z}$

P = nondimensional pressure normalized by $\varrho V_o^2 \varepsilon = P^*/\varrho V_o^2 \varepsilon$.

u = nondimensional radial velocity component normalized by $V_o \varepsilon^{1/2} = v_r^*/V_o \varepsilon^{1/2}$.

v = nondimensional azimuthal velocity component normalized by $V_o = v_\theta^*/V_o$.

V_o = ΩR_o, where Ω is the azimuthal radian angular speed of rotor (Fig. 1).

w = nondimensional axial velocity component normalized by $V_o \varepsilon^{1/2} = v_z^*/V_o \varepsilon^{1/2}$.

θ = azimuthal angular component in radians ($\underline{\hat{\theta}}$ = unit vector). Since radians are nondimensional, there is no superscript (*).

U_m^* = dimensional characteristic of meridional velocities.

Re = $\dfrac{\varrho V_o H \varepsilon^{1/2}}{\mu}$ = Reynolds number defined in this work which is considered to be an $O(1)$ quantity.

R_c^* = dimensional radius of curvature at free surface.

x = $f_{00}(z) + \varepsilon f_{02}(z) + \varepsilon^2 f_{04}(z) + \ldots + F_r^{-1} f_{10}(\theta,z) + \varepsilon^{1/2} F_r^{-1} f_{11}(\theta,z) + \varepsilon F_r^{-1} f_{12}(\theta,z) + F_r^{-2} f_{20}(\theta,z) = f(\theta,z)$ = nondimensional perturbation expansion of free surface.

SYMBOLS (Continued)

$\underset{\sim}{v}$ = $\underset{\sim}{v}_{00}(x,z) + \varepsilon\,\underset{\sim}{v}_{02}(x,z) + \varepsilon^2\underset{\sim}{v}_{04} + \ldots + F_r^{-1}\underset{\sim}{v}_{10}(x,\theta,z) + \varepsilon^{1/2}F_r^{-1}\underset{\sim}{v}_{11}(x,\theta,z) + \varepsilon F_r^{-1}\underset{\sim}{v}_{12}(x,\theta,z) + F_r^{-2}\underset{\sim}{v}_{20}(x,\theta,z) + \ldots$ = nondimensional perturbation expansion series for generalized variable $\underset{\sim}{v}$.

$\dfrac{\varrho v_\theta^{*2}}{r^*}$ = dimensional force term in Navier-Stokes equations.

$\left(\dfrac{\alpha}{R_c}\right) = \dfrac{1}{\varrho V_o^2\varepsilon}\left(\dfrac{\gamma}{R_c^*}\right)$ = nondimensional surface tension term.

$\underset{\sim}{n}$ = $[1 + (f_{00}')^2]^{-1/2}\,[\hat{x} - f_{00}'\hat{z}]$ = unit normal along free surface (Fig. 3).

r^* = $R_o + H\left[f_{00}\left(\dfrac{z^*}{H}\right) + F_r^{-1}\sin\theta F\left(\dfrac{z^*}{H}\right)\right]$ = dimensional free surface to $O(1)$ and $O(F_r^{-1})$.

u = $u_{00} + F_r^{-1}\sin\theta\, U(x,z)$ = nondimensional, nonaxisymmetric asymptotic expansion for the radial velocity component perturbed by gravity to $O(1)$ and $O(F_r^{-1})$.

v = $v_{00} + F_r^{-1}\sin\theta\, V(x,z)$ = nondimensional, nonaxisymmetric asymptotic expansion for the azimuthal velocity component perturbed by gravity to $O(1)$ and $O(F_r^{-1})$.

w = $w_{00} + F_r^{-1}\sin\theta\, W(x,z)$ = nondimensional, nonaxisymmetric asymptotic expansion for the axial velocity component perturbed by gravity to $O(1)$ and $O(F_r^{-1})$.

σ_{ik}^* = $-P^*\delta_{ij} + \mu\left(\dfrac{\partial v_i^*}{\partial x_j^*} + \dfrac{\partial v_j^*}{\partial x_i^*}\right)$ = dimensional stress tensor in an incompressible fluid = $\sigma_{ij}^* = \varrho V_o^2\varepsilon\sigma_{ij}$, where σ_{ij} is the nondimensional stress tensor.

ϱ = density of fluid.

μ = viscosity of fluid.

g = acceleration of gravity.

ε = nondimensional small parameter (i.e., $\varepsilon \ll 1$) = H/R_o used in perturbation expansions.

ABSTRACT

Some designs of liquid metal current collectors in homopolar motors and generators are essentially rotating liquid metal fluids in cylindrical channels with free surfaces and will, at critical rotational speeds, become unstable. An investigation at David Taylor Research Center (DTRC) is being performed to understand the role of gravity in modifying this ejection instability. Some gravitational effects can be theoretically treated by perturbation techniques on the axisymmetric base flow of the liquid metal. This leads to a modification of previously calculated critical current collector ejection values neglecting gravity effects. The purpose of this report is to document the derivation of the mathematical model which determines the perturbation of the liquid metal base flow due to gravitational effects. Since gravity is a small force compared with the centrifugal effects, the base flow solutions can be expanded in inverse powers of the Froude number and modified liquid flow profiles can be determined as a function of the azimuthal angle. This model will be used in later work to theoretically study the effects of gravity on the ejection point of the current collector. A rederivation of the hydrodynamic instability threshold of a liquid metal current collector without gravity effects is presented in an appendix to this report.

ADMINISTRATIVE INFORMATION

This work was a cooperative effort between DTRC and the Department of Mechanical and Industrial Engineering at the University of Illinois, Urbana, Ill. 61801. The work was supported by the DTRC Independent Research Program, Director of Naval Research (OCNR10), and administered by the Research Director (DTRC0113) under Program Element 61152N, Project Number ZR00001, Task Area ZR0230201, Work Unit 1-2712-125, Project Title: "Orientation Effects in Liquid Metal Collectors."

INTRODUCTION

The development of superconductive, direct current homopolar motors and generators used for advanced naval machinery offers opportunities to achieve significant improvements in naval ships. A key feature in the successful operation of these advanced machines is the confinement of the liquid metal used as the sliding electric contact or current collector.

When a fluid with a free surface, e.g., a liquid metal current collector, is rotated in a cylindrical channel (Fig. 1), a critical rotational speed is reached where the liquid is forcibly ejected from the channel. This ejection has been analyzed and correlated with the onset of the Kelvin-Helmholtz flow instability at the free surface resulting from a protective cover gas flowing over the liquid metal. The linear stability analysis performed in previous work by Woo *et al.*[1] consisted of two sequential parts: first, the determination or estimation of the steady-state flow, which is the base solution and results in determining values of the liquid and gas azimuthal velocity V_L and V_G, respectively, at the free surface and, second, the determination of the stability at the free surface. In the theory for determining the ejection point, the inputs are the values of V_L and V_G as a function of Ω, the rotational velocity, and the output is the critical value of Ω (i.e., Ω_{CRIT}) at which ejection occurs because an azimuthal perturbed wave begins to grow with time. This analysis neglects gravitational effects and treats only perfectly axisymmetric base flows and

perturbations so that one azimuthal position is entirely equivalent to every other position. The Appendix presents a clearer, more precise, and somewhat different derivation of the hydrodynamic instability threshold without gravitational effects of a liquid metal current collector than those presented by Woo *et al.*[1] However, the previous results were confirmed by the new derivation.

The objective of this investigation is to determine the role that gravity has on modifying the ejection stability at different orientations of the current collector. This information will allow the design of liquid metal collector systems which will operate in a variety of orientations and, thus, different gravity loadings. The overall gravity investigation being performed at DTRC is divided into experimental and theoretical parts. An experimental apparatus, whose geometry is similar to a current collector and whose orientation with respect to the direction of gravity can be changed, was built and experiments were performed to determine the effects of gravity. This work will be correlated with the present effort in a later report. The other part of the investigation was to determine the effects of gravity on the previously developed theoretical stability analysis by Woo *et al.*[1] Gravitational effects enter into two parts of the stability analysis. In the first part of the previous theory, gravity modifies the base flow solutions by making the flow nonaxisymmetric. The liquid is accelerated or retarded at various azimuthal positions θ, thus affecting the critical ejection angular velocity Ω and defining a preferred angle θ for ejection. In the second part, gravity adds another force to the perturbed wave which destabilizes at the top of the collector but which stabilizes at the bottom of the collector. *The objective of this report is to document the derivation of the mathematical model to determine the effect of gravity on the axisymmetric current collector flow profile.* This mathematical model will be used to develop the theory for the critical ejection rotational velocity with gravity in later works. The model is hydrodynamic and, thus, does not include the magnetic field. This effect will be investigated in later work. Since gravity is a small force compared with centrifugal effects, the base solutions can be expanded in inverse powers of the Froude number and modified liquid profiles can be determined as a function of θ. The surface tension effects are included in the boundary conditions to $O(1)$.

Other sources on current collector magnetohydrodynamic flow theory are available.[2-6]

CURRENT COLLECTOR CONFIGURATION

DISCUSSION OF PROBLEM

Figure 1 shows the geometry of a typical open-channel current collector in a gravitational field. The current collector basically consists of a disk rotating in a cylindrically shaped groove filled with a small amount of liquid metal to provide an electric contact between the disk and the groove. The liquid metal is protected from chemical attack by an inert cover gas, which is also in motion. The rotor has a radius $R_o + L$ and rotates in the azimuthal direction around the z axis with a constant angular velocity Ω. The angle θ is measured from a horizontal radius in the direction of rotation (see Fig. 2). The axial gap width is designated as H, the radial gap as G. In the current collector system, the relatively small dimensions H, G, and L, with respect to R_o have the same order of magnitude designated as $H = O\,(H) = O\,(G) = O\,(L)$. The aspect ratio $\left(\varepsilon = \dfrac{H}{R_o}\right)$ will be a nondimensional parameter, much less than one, that will be of considerable interest in the problem. A Reynolds number shall be defined as $Re = \dfrac{\varrho V_o H \varepsilon^{1/2}}{\mu}$, which is considered to

be of $O(1)$. Here, the equation $V_o = \Omega R_o$ is the velocity of the rotor at the free surface, which ϱ and μ are the density and viscosity of the liquid metal.

For steady-state axisymmetric flow, all derivatives with respect to azimuthal angle θ and time t are zero; however, in this problem the gravitational field perturbs the steady-state axisymmetric flow, and derivatives with respect to θ are retained.

DIMENSIONAL HYDRODYNAMIC EQUATIONS WITH GRAVITATIONAL PERTURBATION

The dimensional hydrodynamic equations for fully developed nonaxisymmetric flow in a gravitational field can be expressed as

$$\varrho\left(v_r{}^* \frac{\partial v_r{}^*}{\partial r^*} + \frac{v_\theta{}^*}{r^*}\frac{\partial v_r{}^*}{\partial \theta} + v_z{}^*\frac{\partial v_r{}^*}{\partial z^*} - \frac{v_\theta{}^{*2}}{r^*}\right) = -\frac{\partial P^*}{\partial r^*} \tag{1}$$

$$- \varrho g \sin\theta + \mu\left(\frac{\partial^2 v_r{}^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial v_r{}^*}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 v_r{}^*}{\partial \theta^2} + \frac{\partial^2 v_r{}^*}{\partial z^{*2}} - \frac{v_r{}^*}{r^{*2}} - \frac{2}{r^{*2}}\frac{\partial v_\theta{}^*}{\partial \theta}\right),$$

$$\varrho\left(v_r{}^* \frac{\partial v_\theta{}^*}{\partial r^*} + \frac{v_\theta{}^*}{r^*}\frac{\partial v_\theta{}^*}{\partial \theta} + v_z{}^*\frac{\partial v_\theta{}^*}{\partial z^*} + \frac{v_r{}^*\, v_\theta{}^*}{r^*}\right) = -\frac{1}{r^*}\frac{\partial P^*}{\partial \theta} - \varrho g \cos\theta \tag{2}$$

$$+ \mu\left(\frac{\partial^2 v_\theta{}^*}{\partial r^*} + \frac{1}{r^*}\frac{\partial v_\theta{}^*}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 v_\theta{}^*}{\partial \theta^2} + \frac{\partial^2 v_\theta{}^*}{\partial z^{*2}} - \frac{v_\theta{}^*}{r^{*2}} + \frac{2}{r^{*2}}\frac{\partial v_r{}^*}{\partial \theta}\right),$$

$$\varrho\left(v_r{}^* \frac{\partial v_z{}^*}{\partial r^*} + \frac{v_\theta{}^*}{r^*}\frac{\partial v_z{}^*}{\partial \theta} + v_z{}^*\frac{\partial v_z{}^*}{\partial z^*}\right) = -\frac{\partial P^*}{\partial z^*} \tag{3}$$

$$+ \mu\left(\frac{\partial^2 v_z{}^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial v_z{}^*}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 v_z{}^*}{\partial \theta^2} + \frac{\partial^2 v_z{}^*}{\partial z^{*2}}\right),$$

and

$$\frac{\partial v_r{}^*}{\partial r^*} + \frac{v_r{}^*}{r^*} + \frac{1}{r^*}\frac{\partial v_\theta{}^*}{\partial \theta} + \frac{\partial v_z{}^*}{\partial z^*} = O, \tag{4}$$

where

$$v_r{}^* = v_r{}^*(r^*,\theta,z^*)\ ,$$

$$v_\theta{}^* = v_\theta{}^*(r^*,\theta,z^*)\ ,$$

and

$$v_z{}^* = v_z{}^*(r^*,\theta,z^*)\ .$$

Equations 1 through 3 are the components of the Navier-Stokes equations in cylindrical coordinates with a gravitational perturbation. Equation 4 is the equation for incompressible flow. Figure 2 shows the gravitational field in the cylindrical coordinate system (r^*,θ). θ is measured from a horizontal radius in the direction of rotation.

The dimensional variables designated by a superscript (*) are defined as follows

v_r^* = dimensional radial velocity component
v_θ^* = dimensional azimuthal velocity component
v_z^* = dimensional axial velocity component
r^* = dimensional radial coordinate ($\hat{r}$ = unit vector)
θ = azimuthal angular component in radians ($\hat{\theta}$ = unit vector). Since radians are nondimensional, there is no superscript.
z^* = dimensional axial coordinate ($\hat{z}$ = unit vector)
P^* = dimensional pressure
ϱ = density of fluid
μ = viscosity of fluid
g = acceleration of gravity.

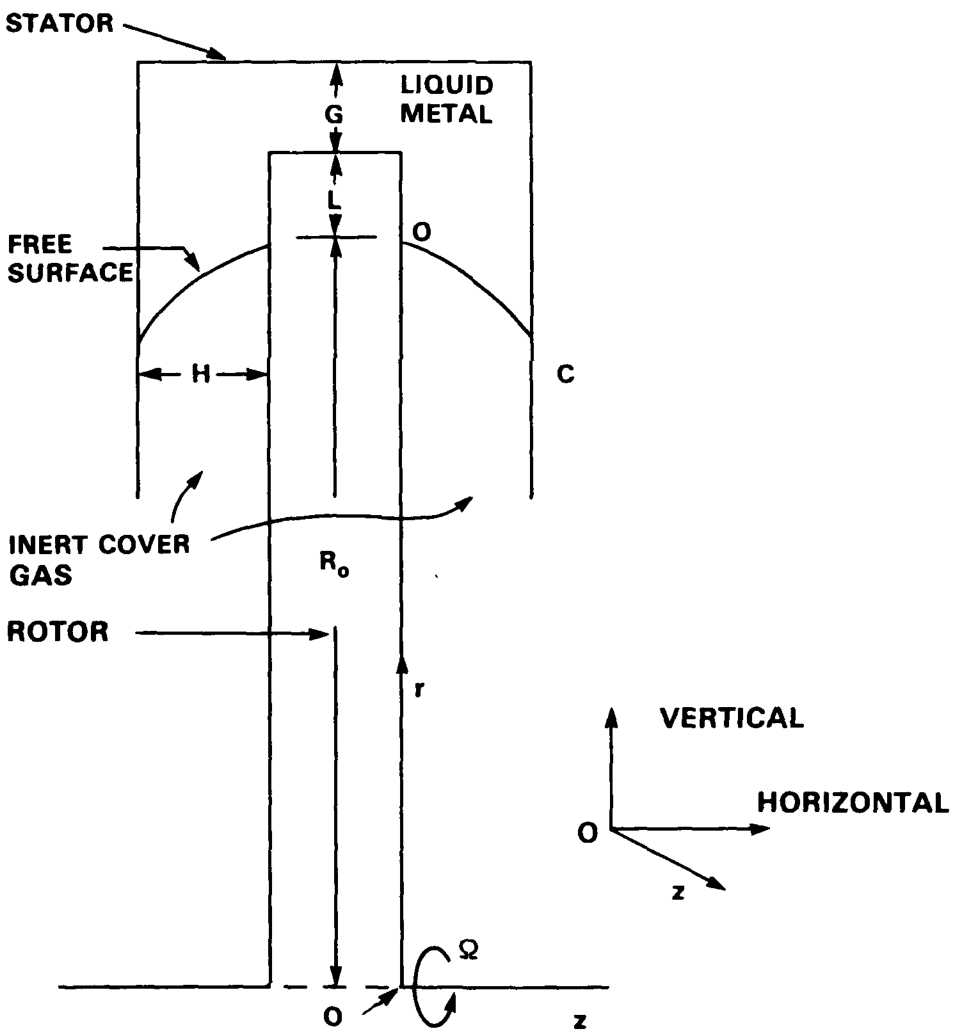

Fig. 1. Geometry and nomenclature for a typical current collector configuration.

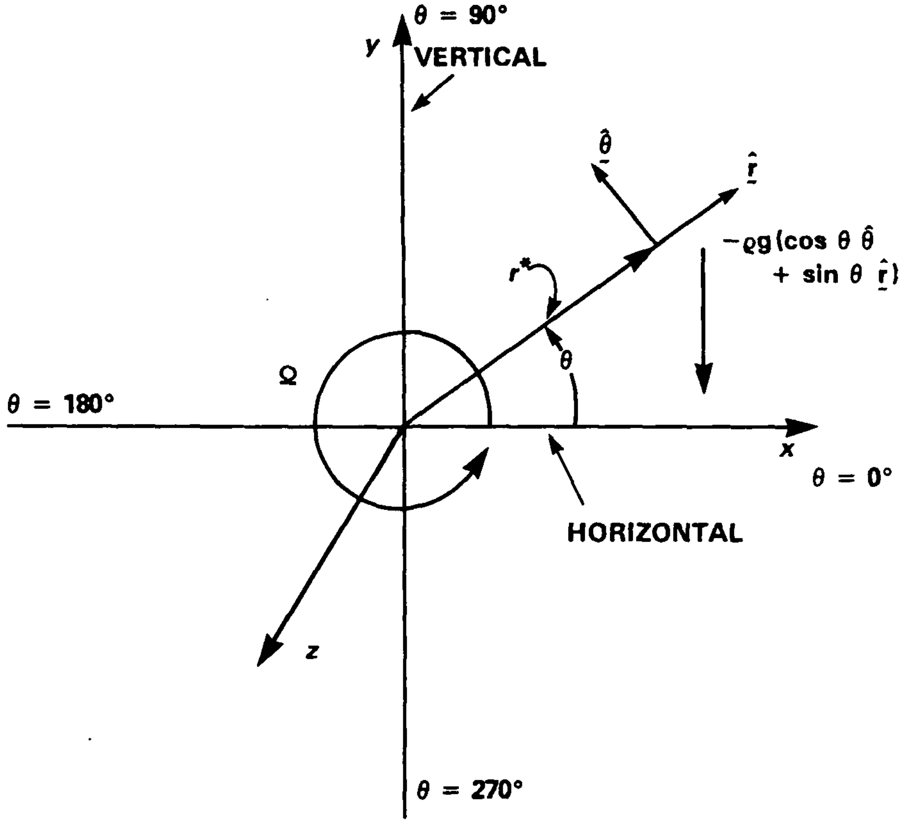

Fig. 2. Gravitational force in r^* and θ components.

In examining Fig. 1, it is noted that the liquid metal is contained on the outer rim of the rotating disk. Therefore, a local radial coordinate $Hx = r^* - R_o$ can be defined at the point O, where the interface between the liquid and gas touches the rotating disk. From this definition of x the following partial differential relationship holds:

$$\frac{\partial}{\partial r^*} = \frac{1}{H}\frac{\partial}{\partial x} \tag{5}$$

CHARACTERISTIC QUANTITIES AND NONDIMENSIONALIZATION

The azimuthal velocity v_θ^* must vary from zero at the stator to $V_o = \Omega r$ at the rotor. Taking R_o to be a characteristic radius, the characteristic azimuthal velocity is $V_o = \Omega R_o$. The meridional flow, (*i.e.*, secondary flow), with the radial velocity v_r^* and the axial velocity v_z^*, is driven by the centrifugal force due to the azimuthal motion $\frac{\varrho v_\theta^{*2}}{r}\,\hat{r}$. The centrifugal force accelerates the meridional circulation until either the inertial term $\varrho \frac{Dv_r^*}{Dt}$ or the viscous term $\mu\left[\nabla^{*2} v_r^* - \frac{v_r^*}{r^{*2}}\right]$ balances the centrifugal force. In this problem there is interest in Reynolds numbers much greater than one ($Re >> 1$). Since the Reynolds

number is defined as the ratio of the inertial force to the viscous force, the inertial term is the first term to balance the centrifugal force $\left(\varrho \frac{Dv_r^*}{Dt^*} = \frac{\varrho v_\theta^{*2}}{r^*} \right)$. This balance determines the characteristic velocity U_m^* for the meridional velocities v_r^* and v_z^*. The following characteristic quantities can be determined:

$$\frac{Dv_r^*}{Dt^*} \sim \frac{v_r^* \partial v_r^*}{\partial r^*} , \; v_r^* = U_m^* , \; \frac{\partial}{\partial r^*} \sim \frac{1}{L} \sim O\left(\frac{1}{H}\right) , \tag{6}$$

and

$$v_\theta^* \sim V_o = \Omega R_o , \; \frac{1}{r^*} \sim \frac{1}{R_o} . \tag{7}$$

Thus, the force balance is the following characteristic quantity:

$$\frac{\varrho U_m^{*\,2}}{H} = \frac{\varrho V_o^2}{R_o} , \tag{8}$$

where

$$U_m^* = \left(\frac{H}{R_o}\right)^{1/2} V_o = \varepsilon^{1/2} V_o . \tag{9}$$

Balancing the pressure gradient $-\frac{\partial P^*}{\partial r}$ against the inertial term one obtains $\varrho V_o^2 \varepsilon$ as the characteristic pressure difference due to the centrifugal force.

Equations 1 through 4 can be made nondimensional by using the following nondimensional variables.

$$v_r^* = V_o \varepsilon^{1/2} u , \tag{10}$$

$$v_\theta^* = V_o \, v , \tag{11}$$

$$v_z^* = V_o \varepsilon^{1/2} w , \tag{12}$$

$$P^* = \varrho V_o^2 \varepsilon P , \tag{13}$$

$$z^* = Hz , \; \frac{\partial}{\partial z^*} = \frac{1}{H}\frac{\partial}{\partial z} , \tag{14}$$

$$r^* = R_o + Hx = \varepsilon^{-1} H(1+\varepsilon x) \tag{15}$$

$$\frac{\partial}{\partial r^*} = \frac{1}{H}\frac{\partial}{\partial x} . \tag{16}$$

The following equations have the following nondimensional form for small values of ε.

$$u\frac{\partial u}{\partial x} + (1+\varepsilon x)^{-1}\varepsilon^{1/2} v \frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - (1+\varepsilon x)^{-1} v^2 \tag{17}$$

$$= -\frac{\partial P}{\partial x} - \left(\frac{gR_o}{V_o^2}\right) \sin\theta + Re^{-1} \left[\frac{\partial^2 u}{\partial x^2} + \varepsilon(1+\varepsilon x)^{-1} \frac{\partial u}{\partial x} \right.$$

$$\left. + \varepsilon^2 (1+\varepsilon x)^{-2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \varepsilon^2 (1+\varepsilon x)^{-2} u - 2\varepsilon^{3/2} (1+\varepsilon x)^{-2} \frac{\partial v}{\partial \theta} \right] ,$$

$$u\frac{\partial v}{\partial x} + \varepsilon^{1/2}(1+\varepsilon x)^{-1} v\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \varepsilon(1+\varepsilon x)^{-1}uv \tag{18}$$

$$= -\varepsilon^{3/2}(1+\varepsilon x)^{-1}\frac{\partial P}{\partial \theta} - \left(\frac{gR_o}{V_o^2}\right)\varepsilon^{1/2}\cos\theta + Re^{-1}\left[\frac{\partial^2 v}{\partial x^2}\right.$$

$$+ \varepsilon(1+\varepsilon x)^{-1}\frac{\partial v}{\partial x} + \varepsilon^2(1+\varepsilon x)^{-2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} - \varepsilon^2(1+\varepsilon x)^{-2}v$$

$$\left. + 2\varepsilon^{5/2}(1+\varepsilon x)^{-2}\frac{\partial u}{\partial \theta}\right] ,$$

$$u\frac{\partial w}{\partial x} + \varepsilon^{1/2}(1+\varepsilon x)^{-1}v\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} \tag{19}$$

$$+ Re^{-1}\left[\frac{\partial^2 w}{\partial x^2} + \varepsilon(1+\varepsilon x)^{-1}\frac{\partial w}{\partial x} + \varepsilon^2(1+\varepsilon x)^{-2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right] ,$$

and

$$\frac{\partial u}{\partial x} + \varepsilon(1+\varepsilon x)^{-1}u + \varepsilon^{1/2}(1+\varepsilon x)^{-1}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 , \tag{20}$$

where

$$u = u(x,\theta,z) ,$$
$$v = v(x,\theta,z) ,$$
$$w = w(x,\theta,z) .$$

The Reynolds number was defined as $Re = \frac{\varrho U_m{}^* H}{\mu} = \frac{\varrho V_o H \varepsilon^{1/2}}{\mu}$ and is considered to be of 0(1) in this problem. The Froude number is defined as $F_r = \frac{V_o^2}{gR_o}$, the inertial force divided by the gravity force. In this problem, the Froude number is much greater than one $(F_r >> 1)$.

BOUNDARY CONDITIONS

At all solid surfaces of the current collector stator and rotor, the nondimensional secondary flow components are zero.

$$u = u(x,\theta,z) = 0 \text{ (at solid surfaces),} \tag{21}$$

and

$$w = w(x,\theta,z) = 0 \text{ (at solid surfaces).} \tag{22}$$

At the stator surface, the nondimensional primary flow component is zero.

Thus,

$$v = v(x,\theta,z) = 0, \text{ at stator,} \tag{23}$$

and at the rotor surface the nondimensional primary flow component is

$$v = v(x,\theta,z) = (1+\varepsilon x), \text{ at rotor.} \tag{24}$$

In the problem described in this report, the nondimensional velocity components are functions of x, θ, and z not x and z, as in the axisymmetric flow case considered earlier by Audet *et al.*[6] This is because the gravitational field perturbation makes the flow in the work discussed in this report nonaxisymmetric.

The boundary conditions at the free surface must be specified. The surface oc in the current collector configuration in Fig. 1 is a surface of separation between two immiscible fluids: the inert cover gas and the liquid metal. The first condition to be satisfied at the free surface requires that the velocities of the fluids in the direction of the unit normal at the free surface be equal to each other and equal to zero. The second condition requires the forces which fluids exert on each other at the free surface be equal and opposite. These conditions are written respectively as

$$v_i n_i = \underset{\sim}{v} \cdot \underset{\sim}{n} = 0 \tag{25}$$

and

$$\sigma_{ij} n_j \text{ (liquid)} + \sigma_{ij} n_j \text{ (inert gas)} = 0. \tag{26}$$

The unit vector for the fluid is along the outward normal vector. The unit normal vector for the gas is in the opposite direction. In order to simplify the boundary conditions surface tension effects, which are important, have been neglected at this point of the work but will be considered later in this report. Neglecting stresses in the inert cover gas at the free surface equation (26) simplifies to

$$\sigma_{ij} n_j = 0, \tag{27}$$

and

$$\sigma_{ij}^* = -P^* \delta_{ij} + \mu \left(\frac{\partial v_i^*}{\partial x_j^*} + \frac{\partial v_j^*}{\partial x_i^*} \right). \tag{28}$$

The stress tensor components σ_{ij} will be made nondimensional later in this report.

PERTURBATION EXPANSIONS OF FUNDAMENTAL VARIABLES

The coupled boundary value partial differential equations and boundary values discussed previously have two small fundamental parameters, ε and F_r^{-1}, for perturbation expansions. An estimate of these parameters for a typical current collector configuration is shown in Fig. 1. If the axial distance is $H = 3 \times 10^{-4}$ meters and the distance along the rotor surface from the z axis to the liquid interface is $R_o = 0.3$ meter, the parameter $\varepsilon = H/R_o = 10^{-3}$ and $\varepsilon^{1/2} \approx 0.0316 = 3.16 \times 10^{-2}$. For a typical speed of 3,000 revolutions per minute or 314.2 radians per second, the velocity $\Omega_o R_o = 94.25$ meters per second. The Froude number $F_r = V_o^2/gR_o$ is calculated to be 3,021.3, F_r^{-1} equals 3.31×10^{-4} and $F_r^{-1} << 1$. The Froude number is a nondimensional parameter. The Reynolds number $Re = \dfrac{\varrho V_o H \varepsilon^{1/2}}{\mu}$ is considered as an $0(1)$ parameter and does not enter the asymptotic analysis. In an asymptotic expansion, $\varepsilon^{1/2}$ and F_r^{-1} are considered to approach zero.

ASYMPTOTIC EXPANSION

The limits as $\varepsilon^{1/2}$ and $F_r^{-1} \rightarrow 0$ are considered *independently*, so a double asymptotic expansion for the velocity $\hat{\underset{\sim}{v}}$ is required. Thus,

$$\underset{\sim}{v} = \underset{\sim}{v}_{oo} + \varepsilon^{1/2} \underset{\sim}{v}_{01} + \varepsilon \underset{\sim}{v}_{02} + \varepsilon^{3/2} \underset{\sim}{v}_{03} + \ldots \tag{29}$$

$$+ F_r^{-1}\underset{\sim}{v}_{10} + \varepsilon^{1/2}F_r^{-1}\underset{\sim}{v}_{11} + \varepsilon F_r^{-1}\underset{\sim}{v}_{12} + \varepsilon^{3/2}F_r^{-1}\underset{\sim}{v}_{13} + \ldots$$

$$+ F_r^{-2}\underset{\sim}{v}_{20} + \varepsilon^{1/2}F_r^{-2}\underset{\sim}{v}_{21} + \varepsilon F_r^{-2}\underset{\sim}{v}_{22} + \ldots$$

$$+ F_r^{-3}\underset{\sim}{v}_{30} + \varepsilon^{1/2}F_r^{-3}\underset{\sim}{v}_{31} + \ldots$$

$$+ F_r^{-4}\underset{\sim}{v}_{40} + \ldots$$

$$+ \ldots .$$

This explicit form applies for each component of velocity u, v, and w and for the pressure P.

Gravitational effects first enter the expansion in Eq. 29 on the second line. Each term in the first line v_{oj}, for $j = 0, 1, 2, \ldots$ is independent of gravitational effects.

Without gravitational effects, the flow is perfectly axisymmetric. Therefore, $\underset{\sim}{v}_{00}$, $\underset{\sim}{v}_{01}$, $\underset{\sim}{v}_{02}$, $\underset{\sim}{v}_{03}$, etc., are all independent of θ. In addition, the axisymmetric version of the governing equations does not involve $\varepsilon^{1/2}$, only ε. For the first row of this double series, the only small parameter is ε, not $\varepsilon^{1/2}$, so that $\underset{\sim}{v}_{01} \equiv 0$, $\underset{\sim}{v}_{03} \equiv 0$, $\underset{\sim}{v}_{05} \equiv 0$, etc. Therefore,

$$\underset{\sim}{v} = \underset{\sim}{v}_{00}(x,z) + \varepsilon \underset{\sim}{v}_{02}(x,z) + \varepsilon^2 \underset{\sim}{v}_{04}(x,z) + \ldots \tag{30}$$

$$+ F_r^{-1}\underset{\sim}{v}_{10}(x,\theta,z) + \varepsilon^{1/2}F_r^{-1}\underset{\sim}{v}_{11}(x,\theta,z) + \varepsilon F_r^{-1}\underset{\sim}{v}_{12}(x,\theta,z)$$

$$+ F_r^{-2}\underset{\sim}{v}_{20}(x,\theta,z) + \ldots .$$

Only two terms in this series, $\underset{\sim}{v}_{00}(x,z)$ and $\underset{\sim}{v}_{10}(x,\theta,z)$, where $\underset{\sim}{v}_{00}(x,z)$ is the axisymmetric base solution will be computed. The base solution neglects gravitational effects and assumes $\varepsilon << 1$, while $\underset{\sim}{v}_{10}(x,\theta,z)$ is the first perturbation of the base solution due to gravitational effects. $\underset{\sim}{v}_{02}$, will be ignored even though ε is comparable to F_r^{-1}. Mathematically, this double series is valid, as $\varepsilon \rightarrow 0$ and $F_r^{-1} \rightarrow 0$, independently. Therefore, the expressions for $\underset{\sim}{v}_{10}$ and $\underset{\sim}{v}_{02}$ can be independently derived. Each term in the series depends only on terms above and to the left of it. Physically, $\varepsilon \underset{\sim}{v}_{02}(x,z)$ may be a larger perturbation than $F_r^{-1}\underset{\sim}{v}_{10}(x,\theta,z)$, but the former is still axisymmetric, while the latter represents the first deviation from axisymmetry. The location of the ejection point is determined by the deviation from axisymmetry, regardless of how small, because this deviation determines which point becomes unstable slightly before another point.

Substituting $u = u_{00}(x,z)$, $v = v_{00}(x,z)$, $w = w_{00}(x,z)$, and $P = P_{00}(x,z)$ Eqs. 17 through 19 and the boundary conditions result in the following system of coupled base equations:

$$u_{00}\frac{\partial u_{00}}{\partial x} + w_{00}\frac{\partial u_{00}}{\partial z} - v_{00}^2 = -\frac{\partial P_{00}}{\partial x} + Re^{-1}\left[\frac{\partial^2 u_{00}}{\partial x^2} + \frac{\partial^2 u_{00}}{\partial z^2}\right], \tag{31}$$

$$u_{00}\frac{\partial v_{00}}{\partial x} + w_{00}\frac{\partial v_{00}}{\partial z} = Re^{-1}\left[\frac{\partial^2 v_{00}}{\partial x^2} + \frac{\partial^2 v_{00}}{\partial z^2}\right], \tag{32}$$

$$u_{00}\frac{\partial w_{00}}{\partial x} + w_{00}\frac{\partial w_{00}}{\partial z} = -\frac{\partial P_{00}}{\partial z} + Re^{-1}\left[\frac{\partial^2 w_{00}}{\partial x^2} + \frac{\partial^2 w_{00}}{\partial z^2}\right] , \tag{33}$$

and

$$\frac{\partial u_{00}}{\partial x} + \frac{\partial w_{00}}{\partial z} = 0 . \tag{34}$$

This is the same result obtained by Audet *et al.*,[6] except they made the pressure p^* nondimensional by $P^* = \frac{\varepsilon^{1/2} V_o \mu P}{H}$, which differs from the factor which makes the pressure nondimensional, $\varrho v_o^2 \varepsilon$, (i.e. $P^* = \varrho v_o^2 \varepsilon P$), by a factor of the Reynolds number. The work of Audet *et al.*[6] is fully axisymmetric and does not include the effects of gravity.

BOUNDARY CONDITIONS

The governing nonlinear partial differential equations determine $u_{00}(x,z)$, $v_{00}(x,z)$, $w_{00}(x,z)$, and $P_{00}(x,z)$ for specified boundary conditions. The following boundary conditions are specified at the current collector surfaces. At all solid surfaces both rotor and stator are

$$u_{00}(x,z) = w_{00}(x,z) = 0 . \tag{35}$$

$v_{00}(x,z) = 0$ at the stator surfaces, and $v_{00}(x,z) = 1$ at all rotor surfaces. For the boundary conditions at the free surfaces, the components of the stress tensor are required and are normalized by $\varrho V_o^2 \varepsilon$. With Eqs. 17 through 20 made nondimensional and the coordinate system being changed from r,θ, and z to x,θ, and z

$$\sigma_{ij}^* = \varrho V_o^2 \varepsilon \sigma_{ij} . \tag{36}$$

The components of the dimensional stress tensor σ_{ij}^* from dimensional cylindrical coordinates to nondimensional coordinates σ_{ij} are:

$$\sigma_{rr}^* = -P^* + 2\mu\left(\frac{\partial v_r^*}{\partial r^*}\right) \rightarrow \sigma_{xx} = -P + 2Re^{-1}\frac{\partial u}{\partial x} , \tag{37}$$

$$\sigma_{\theta\theta}^* = -P^* + 2\mu\left(\frac{1}{r^*}\frac{\partial v_\theta^*}{\partial \theta^*} + \frac{v_r^*}{r^*}\right) \tag{38}$$

$$\rightarrow \sigma_{\theta\theta} = -P + 2Re^{-1}\varepsilon^{1/2}(1+\varepsilon x)^{-1}\left(\frac{\partial v}{\partial \theta} + \varepsilon^{1/2} u\right) ,$$

$$\sigma_{zz}^* = -P^* + 2\mu\frac{\partial v_z^*}{\partial z^*} \rightarrow \sigma_{zz} = -P + 2Re^{-1}\frac{\partial w}{\partial z} , \tag{39}$$

$$\sigma_{r\theta}^* = \sigma_{\theta r}^* = \mu\left(\frac{1}{r^*}\frac{\partial v_r^*}{\partial \theta} + \frac{\partial v_\theta^*}{\partial r^*} - \frac{v_\theta^*}{r^*}\right) \tag{40}$$

$$\rightarrow \sigma_{x\theta} = \sigma_{\theta x} = Re^{-1}\left[\varepsilon^{-1/2}\left(\frac{\partial v}{\partial x}\right) + \varepsilon^{1/2}(1+\varepsilon x)^{-1}\left(\varepsilon^{1/2}\frac{\partial u}{\partial \theta} - v\right)\right] ,$$

$$\sigma_{zr}^* = \sigma_{rz}^* = \mu\left(\frac{\partial v_z^*}{\partial r^*} + \frac{\partial v_r^*}{\partial z^*}\right) \tag{41}$$

$$\rightarrow \sigma_{xz} = \sigma_{zx} = Re^{-1}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right],$$

and

$$\sigma_{z\theta}^* = \sigma_{\theta z}^* = \mu\left(\frac{\partial v_\theta^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial v_z^*}{\partial \theta}\right) \tag{42}$$

$$\rightarrow \sigma_{\theta z} = \sigma_{z\theta} = Re^{-1}\left[\varepsilon^{-1/2}\frac{\partial v}{\partial z} + \varepsilon^{1/2}(1+\varepsilon x)^{-1}\frac{\partial w}{\partial \theta}\right].$$

In order to derive the stresses on the free surface, the free surface location must be known. However, the free surface $x = f(\theta,z)$ is another unknown of the problem. Therefore, the free surface will be expanded in a perturbation series. Thus,

$$\begin{aligned} x = {} & f_{00}(z) + \varepsilon f_{02}(z) + \varepsilon^2 f_{04}(z) + \ldots \\ & + F_r^{-1} f_{10}(\theta,z) + \varepsilon^{1/2} F_r^{-1} f_{11}(\theta,z) + \varepsilon f_r^{-1} F_{12}(\theta,z) + \ldots \\ & + F_r^{-2} f_{20}(\theta,z) + \ldots \quad = f(\theta,z) . \end{aligned} \tag{43}$$

$P(x,\theta,z)$, $u(x,\theta,z)$, $v(x,\theta,z)$, and $w(x,\theta,z)$ are expanded in the same form as $x = f(\theta,z)$. All of these unknowns are governed by the partial differential equations (Eqs. 17 through 20), the boundary conditions at the rotor and stator surfaces and the four boundary conditions at the free surface (normal velocity and three components of surface traction equal zero). The solution for each unknown depends on the parameters ε, F_r^{-1}, Re, and geometry. Each variable can be replaced by its double asymptotic expansion in ε and F_r^{-1}, where the coefficient functions still depend on Re and the geometry but are independent of ε and F_r^{-1}. Like the other variables, $f(\theta,z)$ has a first row which is independent of θ (axisymmetric) and involves only ε, not $\varepsilon^{1/2}$.

The boundary conditions on u_{00}, v_{00}, w_{00}, and P_{00} at $x = f_{00}(z)$ are:

$$n_x u + n_z w = 0, \tag{44}$$

and

$$n_x \sigma_{ix} + n_z \sigma_{iz} = \left(\frac{\alpha}{R_c}\right) n_i, \text{ for } i = (x,\theta,z), \tag{45}$$

where the nondimensional surface tension term $\left(\frac{\alpha}{R_c}\right) = \frac{1}{\varrho V_o^2 \varepsilon}\left(\frac{\gamma}{R_c^*}\right)$ and R_c^* equals the dimensional radius of curvature along the free surface. R_c^* is considered positive when the surface is convex in the gas phase. Here γ is the surface tension and $\alpha = \gamma/\varrho V_o^2 \varepsilon H$ is a dimensionless surface tension coefficient. The surface tension term is necessary for numerical stability for numerical calculations to zero order. The dimensional radius of curvature is expressed as

$$\frac{1}{HR_c} = \frac{1}{R_c^*} = \frac{-f_{00}^{*\prime\prime}}{[1 + (f_{00}^{*\prime})^2]^{3/2}} . \tag{46}$$

The unit normal along the free surface $\underset{\sim}{n}$ is defined as

$$\underset{\sim}{n} = [1+(f_{00}')^2]^{-1/2}\,[\hat{x} - f_{00}'\,\hat{z}] , \tag{47}$$

where

$$n_x = [1 + (f'_{00})^2]^{-1/2},\ n_\theta = 0 \tag{48}$$

and

$$n_z = -f'_{00}\,[1 + (f'_{00})^2]^{-1/2}, \tag{49}$$

$$\hat{\underset{\sim}{x}} = \text{radial unit vector}, \tag{50}$$

$$\hat{\underset{\sim}{z}} = \text{axial unit vector}, \tag{51}$$

and

$$f'_{00} = \frac{df_{00}(z)}{dz}\ . \tag{52}$$

Therefore, the base boundary solutions at the free surface are at $x = f_{00}(z)$ are as follows:

$$u_{00} = f'_{00}\, w_{00}\ , \tag{53}$$

$$\frac{\partial v_{00}}{\partial x} = f'_{00}\,\frac{\partial v_{00}}{\partial z}\ , \tag{54}$$

$$-\ Re\, P_{00} - Re\left(\frac{\alpha}{R_c}\right) + 2\,\frac{\partial u_{00}}{\partial x} = f'_{00}\left(\frac{\partial u_{00}}{\partial z} + \frac{\partial w_{00}}{\partial x}\right), \tag{55}$$

and

$$f'_{00}\, Re P_{00} + f'_{00}\, Re\left(\frac{\alpha}{R_c}\right) + \frac{\partial u_{00}}{\partial z} + \frac{\partial w_{00}}{\partial x} = 2f'_{00}\,\frac{\partial w_{00}}{\partial z}\ . \tag{56}$$

MODIFICATION AND DISCUSSION OF BOUNDARY CONDITIONS

Traditionally, two linear combinations of the last two equations are taken. The first combination eliminates the pressure P_{00}:

$$f'_{00}(z)\ [\text{Eq. } 59] + [\text{Eq. } 55]. \tag{57}$$

Thus,

$$2f'_{00}\,\frac{\partial u_{00}}{\partial x} + \frac{\partial u_{00}}{\partial z} + \frac{\partial w_{00}}{\partial x} = (f'_{00})^2\,\frac{\partial u_{00}}{\partial z} + (f'_{00})^2\,\frac{\partial w_{00}}{\partial x} + 2f'_{00}\,\frac{\partial w_{00}}{\partial z}\ . \tag{58}$$

At $x = f_{00}(z)$, the normal component of the velocity v_n equals zero. v_n can be differentiated along the surface using the chain rule, and the result can be used to simplify the condition (Eq. 58) further. The condition on v_n is

$$u_{00}(f_{00}(z),z) = f'_{00}(z)\ w_{00}(f_{00}(z),\ z). \tag{59}$$

Differentiating with respect to z, the following equation is obtained:

$$f'_{00}\,\frac{\partial u_{00}(f_{00},z)}{\partial x} + \frac{\partial u_{00}(f_{00},z)}{\partial z} = f''_{00}\ w_{00}(f_{00},z) + (f'_{00})^2\,\frac{\partial w_{00}(f'_{00},z)}{\partial x} + f'_{00}\,\frac{\partial w_{00}(f_{00},z)}{\partial z}\ . \tag{60}$$

Subtracting Eq. 60 from Eq. 58, the following boundary condition is obtained:

$$f_{00}' \left[\frac{\partial u_{00}}{\partial x} - f_{00}' \frac{\partial u_{00}}{\partial z} \right] + \left[\frac{\partial w_{00}}{\partial x} - f_{00}' \frac{\partial w_{00}}{\partial z} \right] \tag{61}$$

$$+ f_{00}'' w_{00} = 0 \,, \qquad \text{at } x = f_{00}(z) \,.$$

Equation 61 represents the stress condition, $\sigma_{nt} = 0$, where (n,θ,t) are orthogonal curvilinear coordinates; the unit normal $\hat{t}$ is tangent to the surface $x = f_{00}(z)$ in each θ = constant plane, as shown in Fig. 3. If the surface is a straight line in each θ = constant plane, then

$$\sigma_{nt} = Re^{-1} \left(\frac{\partial v_t}{\partial n} + \frac{\partial v_n}{\partial t} \right) = 0 \,. \tag{62}$$

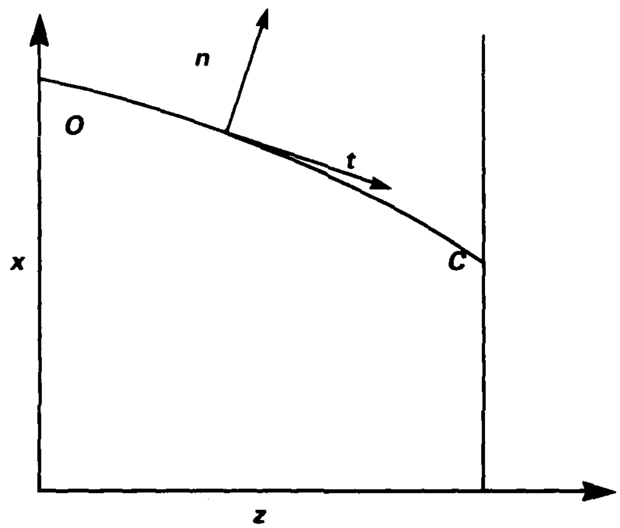

Fig. 3. Free surface $x = f_{00}(z)$ with normal vector n and tangential vector t.

Since $v_n = 0$ here, $\frac{\partial v_n}{\partial t} = 0$, which is equivalent to subtracting Eq. 60 from Eq. 58. The boundary condition for a straight line surface is

$$\frac{\partial v_t}{\partial n} = [1 + (f_{00}')^2]^{-1} \left[\frac{\partial}{\partial x} - f_{00}' \frac{\partial}{\partial z} \right] (f_{00}' u_{00} + w_{00}), \tag{63}$$

which reduces to the first four terms in Eq. 61 for a straight line surface with f_{00}' = constant and $f_{00}'' = 0$. The fifth term in Eq. 61, $f_{00}'' w_{00}$, represents the effect of surface curvature in each θ = constant plane. At each point on the free surface, polar coordinates are taken in a θ = constant plane with the origin at the local center of curvature. In these polar coordinates, (R,α), the surface is at $R = R_c$ = local radius of curvature, and α changes along the surface. Now the equation becomes

$$\sigma_{R\alpha} = Re^{-1} \left[\frac{1}{R_c} \frac{\partial v_R}{\partial \alpha} + \frac{\partial v_\alpha}{\partial R} - \frac{v_\alpha}{R_c} \right] = 0, \tag{64}$$

where

$$\frac{1}{R}\frac{\partial v_R}{\partial \alpha} = \frac{\partial v_n}{\partial t} = 0; \quad \frac{\partial v_\alpha}{\partial R} = \frac{\partial v_t}{\partial n}, \tag{65}$$

and the final new term, $-\dfrac{v_\alpha}{R_c}$, represents the extra term due to curvature. Since $u_{00} = f_{00}' w_{00}$ at the surface, the following equation is obtained:

$$\begin{aligned} v_\alpha = v_t &= [1 + (f_{00}')^2]^{-1/2} (f_{00}' u_{00} + w_{00}) \\ &= [1 + (f_{00}')^2]^{1/2} w_{00} . \end{aligned} \tag{66}$$

The local radius of curvature is provided by

$$\frac{1}{R_c} = - f_{00}'' [1 + (f_{00}')^2]^{-3/2}, \tag{67}$$

so that the extra term due to surface curvature is

$$-\frac{v_\alpha}{R_c} = f_{00}'' [1 + (f_{00}')^2]^{-1} w_{00}. \tag{68}$$

The sum of the straight line surface condition (Eq. 63) and the extra term due to surface curvature (Eq. 68) provides the boundary condition of equation (61).

The other linear combinations of the two equations involving P_{00} are obtained from Eqs. 55 and 56:

f_{00}' (Eq. 56) − (Eq. 55).

Thus,

$$\begin{aligned} P_{00} = \left(-\frac{\alpha}{R_c} \right) &+ 2Re^{-1} [1 + (f_{00}')^2]^{-1} \left\{ \frac{\partial u_{00}}{\partial x} - f_{00}' \frac{\partial u_{00}}{\partial z} \right. \\ &\left. - f_{00}' \left[\frac{\partial w_{00}}{\partial x} - f_{00}' \frac{\partial w_{00}}{\partial z} \right] \right\} , \text{ at } x = f_{00}(z) . \end{aligned} \tag{69}$$

This equation represents the boundary condition normal to the surface

$$\sigma_{nn} = - P_{00} - \left(\frac{\alpha}{R_c} \right) + 2Re^{-1} \frac{\partial v_n}{\partial n} = 0 , \tag{70}$$

where

$$\frac{\partial}{\partial n} = [1 + (f_{00}')^2]^{-1/2} \left[\frac{\partial}{\partial x} - f_{00}' \frac{\partial}{\partial z} \right] , \tag{71}$$

and

$$v_n = [1 + (f_{00}')^2]^{-1/2} [u_{00} - f_{00}' w_{00}] . \tag{72}$$

Curvature terms do not enter into this equation. In the following work, Eqs. 61 and 70 shall be used for the boundary conditions.

The boundary value problem for P_{00} (x,z), u_{00} (x,z), v_{00} (x,z), w_{00} (x,z), and f_{00} (z) is the nonlinear problem currently being solved numerically by Audet *et al.*,[6] for the DTRC current collector program for various high values of the Reynolds number. Audet *et al.*[6] have developed a numerical scheme for the solution of the zeroth order Navier-Stokes equations in order to calculate the equilibrium flow distribution in the liquid metal region

of a current collector (Fig. 1). The computer program calculates problems for Reynolds numbers of up to 1,000. The solution of the finite difference equations is based on a Newton-Raphson method. A mathematical method is being developed to calculate the free surface shape from the boundary conditions. Audet *et al.*[6] are in the process of calculating a variety of velocity profiles for a number of current collector geometries; henceforth, it can be assumed that this base solution is known and the first perturbation can be considered due to gravitational effects, i.e., $P_{10}\,(x,\theta,z)$, $u_{10}\,(x,\theta,z)$, $v_{10}\,(x,\theta,z)$, $w_{10}\,(x,\theta,z)$, and $f_{10}\,(\theta,z)$,

GRAVITATIONAL PERTURBATION EXPANSION

In Eqs. 17 through 20, the θ derivatives appear as $\varepsilon^{1/2}\,\dfrac{\partial}{\partial\theta}$ or $\varepsilon^{2}\,\dfrac{\partial^2}{\partial\theta^2}$. The first term with θ variation is $F_r^{-1}\underset{\sim}{v}_{10}$, so that the θ derivative terms are all $\varepsilon^{1/2}F_r^{-1}$ or $\varepsilon^{2}F_r^{-1}$, which are negligible compared to the F_r^{-1} term. Therefore, θ enters through the gravitational terms, but no θ derivatives enter so that the solution can be obtained in each θ = constant plane independently of all other θ = constant planes.

The two gravitational terms are $-F_r^{-1}\sin\theta$ in the radial equation and $-F_r^{-1}\,\varepsilon^{1/2}\cos\theta$ in the azimuthal equation. The former drives the perturbation; the latter term is negligible in this problem.

Therefore, the first gravitational perturbation terms can be written as:

$$u_{10}\,(x,\theta,z) = \sin\theta\; U\,(x,z)\;, \tag{73}$$

$$v_{10}\,(x,\theta,z) = \sin\theta\; V\,(x,z)\;, \tag{74}$$

$$w_{10}\,(x,\theta,z) = \sin\theta\; W\,(x,z)\;, \tag{75}$$

$$f_{10}\,(\theta,z) = \sin\theta\; F\,(z)\;. \tag{76}$$

The effects of gravity are maximum and opposite at the top ($\theta = 90°$) and bottom ($\theta = 270°$); they are negligible at $\theta = 0°$ and $\theta = 180°$. When the asymptotic expansions

$$\underset{\sim}{v} = \underset{\sim}{v}_{00}\,(x,z) + F_r^{-1}\sin\theta\;\underset{\sim}{V}\,(x,z) \tag{77}$$

are introduced into Eqs. 17 through 20 and the terms $0(\varepsilon)$, $0\,(F_r^{-1}\varepsilon^{1/2})$, and $0\,(F_r^{-2})$ are neglected and the equations governing u_{00}, v_{00}, w_{00}, and P_{00} are subtracted and the remainder is multiplied by F_r (csc θ), then a set of linear equations is obtained governing U, V, W, and P, which have variable coefficients involving the known solutions u_{00}, v_{00}, and w_{00}.

Thus,

$$u_{00}\,\frac{\partial U}{\partial x} + \frac{\partial u_{00}}{\partial x}\,U + w_{00}\,\frac{\partial U}{\partial z} + \frac{\partial u_{00}}{\partial z}\,W \tag{78}$$

$$-\;2\,v_{00}\,V = -\,\frac{\partial P}{\partial x} - 1 + Re^{-1}\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right]\;.$$

The -1 in Eq. 78 is derived from $-\left(\dfrac{gR_o}{V_o^2}\right)\sin\theta$ and is the only inhomogeneous, driving term in the boundary value problem; the known functions come first:

$$u_{00} \frac{\partial V}{\partial x} + \frac{\partial v_{00}}{\partial x} U + w_{00} \frac{\partial V}{\partial z} + \frac{\partial v_{00}}{\partial z} W \tag{79}$$

$$= Re^{-1} \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} \right] ,$$

$$u_{00} \frac{\partial W}{\partial x} + \frac{\partial w_{00}}{\partial x} U + w_{00} \frac{\partial W}{\partial z} + \frac{\partial w_{00}}{\partial z} W = - \frac{\partial P}{\partial z} \tag{80}$$

$$+ Re^{-1} \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} \right] ,$$

and

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 , \tag{81}$$

where

$$U = U(x,z) ,$$

$$V = V(x,z) ,$$

and

$$W = W(x,z) .$$

The following boundary conditions apply to Eqs. 79 through 81. Equation 82 is true at all solid surfaces, both stator and rotor.

Thus,

$$U = 0,\ V = 0, \text{ and } W = 0. \tag{82}$$

At the free surface, the following overall conditions remain:

$$v_i n_i = 0, \text{ where } i = x, \theta, z , \tag{83}$$

and

$$\sigma_{ij} n_i = \left(\frac{\alpha}{R_c} \right) . \tag{84}$$

The four boundary conditions expressed in Eqs. 83 and 84 will be evaluated in the next section to 0 (F_r^{-1}) to show the effects of gravity. The effects of surface tension are neglected at 0 (F_r^{-1}) because the surface tension effects are not needed for numerical stability as in the case of the zeroth order.

GRAVITATIONAL PERTURBATION BOUNDARY CONDITIONS

UNIT NORMAL TO FREE SURFACE

The dimensional coordinate at the free surface r^* is defined to 0(1) and 0 (F_r^{-1}). Thus,

$$r^* = R_o + H \left[f_{00} \left(\frac{z^*}{H} \right) + F_r^{-1} \sin\theta\, F \left(\frac{z^*}{H} \right) \right] , \tag{85}$$

where

$$x = f_{00}(z) + F_r^{-1} \sin\theta\, F(z) , \tag{86}$$

and

$$r^* = R_o + Hx ,\ zH = z^*. \tag{87}$$

The function ξ can be defined as

$$\xi\,(r^*,\theta,z^*) = r^* - R_o - H\left[f_{00}\left(\frac{z^*}{H}\right) + F_r^{-1}\sin\theta F\left(\frac{z^*}{H}\right)\right] \tag{88}$$

where $E = 0$ at the free surface perturbed with $0\,(F_r^{-1})$ gravity terms. From the definition of the gradient vector $\underset{\sim}{\nabla}$, the normal to the free surface is

$$\underset{\sim}{\nabla}\xi\Big|_{\xi=0} = \frac{\partial\xi}{\partial r^*}\Big|_{\xi=0}\hat{\underset{\sim}{r}} + \frac{1}{r^*}\frac{\partial\xi}{\partial\theta}\Big|_{\xi=0}\hat{\underset{\sim}{\theta}} + \frac{\partial\xi}{\partial z^*}\Big|_{\xi=0}\hat{\underset{\sim}{z}} \tag{89}$$

$$= \hat{\underset{\sim}{r}} - \varepsilon\,(1+\varepsilon x)^{-1}F_r^{-1}\cos\theta\,F(z)\,\hat{\underset{\sim}{\theta}}$$

$$-\,[\,f_{00}'(z) + F_r^{-1}\sin\theta\,F'(z)]\,\hat{\underset{\sim}{z}}\,.$$

The unit normal is then expressed as

$$\underset{\sim}{n} = \underset{\sim}{\nabla}\xi\Big|_{\xi=0}\Big/\left|\,\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right| = \left(\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right)\left(\left|\,\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right|\right)^{-1}, \tag{90}$$

where the components n_x, n_θ, and n_z are:

$$n_x = \left(\left|\,\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right|\right)^{-1}, \tag{91}$$

$$n_\theta = -\varepsilon\,(1+\varepsilon x)^{-1}F_r^{-1}\left|\,\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right|^{-1}\cos\theta\,F(z)\,, \tag{92}$$

and

$$n_z = -\left|\,\underset{\sim}{\nabla}\xi\Big|_{\xi=0}\right|^{-1}\left[f_{00}'(z) + F_r^{-1}\sin\theta\,F'(z)\right]. \tag{93}$$

DERIVED GRAVITY BOUNDARY CONDITIONS

The boundary condition Eq. 83 will be evaluated and can be expressed as

$$un_x + vn_\theta + wn_z = 0 \tag{94}$$

at the free surface. Substituting n_x, n_θ, and n_z from Eqs. 91 through 93 and keeping terms to $0(1)$ and $0(F_r^{-1})$, results in the expression

$$u = [f_{00}'(z) + F_r^{-1}\sin\theta\,F'(z)]w\,,$$

$$\text{at } x = f_{00}(z) + F_r^{-1}\sin\theta\,F(z)\,, \tag{96}$$

substituting $u = u_{00}\,(x,z) + F_r^{-1}\sin\theta\,U\,(x,z)$ and $w = w_{00}\,(x,z) + F_r^{-1}\sin\theta\,W\,(x,z)$ and using Eq. 96 for x at the free surface provides

$$u_{00}\,(f_{00}(z) + F_r^{-1}\sin\theta\,F(z),z) \tag{97}$$

$$+\,F_r^{-1}\sin\theta\,U\,(f_{00}\,(z) + F_r^{-1}\sin\theta\,F\,(z),\,z)$$

$$= [f_{00}'(z) + F_r^{-1} \sin\theta\, f_{00}'(z)]$$
$$\times \{w_{00}(f_{00}(z) + F_r^{-1} \sin\theta\, F(z), z)$$
$$+ F_r^{-1} \sin\theta\, W(f_{00}(z) + F_r^{-1} \sin\theta\, F(z), z)\} .$$

Careful examination of Eq. 97 reveals that it is not an asymptotic expansion. A true asymptotic expansion must consist of powers or functions of the small parameter or parameters (F_r^{-1}) times coefficient functions which are independent of F_r^{-1}. Terms such as $u_{00}(f_{00}(z) + F_r^{-1} \sin\theta\, F(z), z)$ depend on F_r^{-1}.

Therefore, the Taylor series expansion for u_{00} is introduced, which is

$$u_{00}(f_{00}(z) + F_r^{-1} \sin\theta\, \mathrm{F}(z), z) = u_{00}(f_{00}(z), z) \tag{98}$$
$$+ F_r^{-1} \sin\theta\, F(z) \frac{\partial u_{00}(f_{00}(z), z)}{\partial x}$$
$$+ \tfrac{1}{2} F_r^{-2} \sin^2\theta\, [F(z)]^2 \frac{\partial^2 u_{00}(f_{00}(z), z)}{\partial x^2} + \ldots$$

Also, analogous series for w_{00}, U, and W are developed. These Taylor series expansions are substituted in Eq. 97, and terms for 0 (F_r^{-2}) are neglected to obtain:

$$u_{00}(f_{00}(z), z) + F_r^{-1} \sin\theta \left[F(z) \frac{\partial u_{00}(f_{00}(z), z)}{\partial x} \right. \tag{99}$$
$$\left. + U(f_{00}(z), z) \right] = f_{00}'(z)\, w_{00}(f_{00}(z), z) + F_r^{-1} \sin\theta$$
$$\times \left[F'(z) w_{00}(f_{00}(z), z) + f_{00}'(z) F(z) \frac{\partial w_{00}(f_{00}(z), z)}{\partial x} \right.$$
$$\left. + f_{00}'(z) W(f_{00}(z), z) \right] .$$

Subtracting Eq. 53, the $O(1)$ part of the boundary condition, from Eq. 97 and multiplying by $F_r \csc\theta$ results in the gravity boundary condition at 0 (F_r^{-1}). Thus,

$$\frac{\partial u_{00}(f_{00}(z), z)}{\partial x} \mathrm{F}(z) + U(f_{00}(z), z) \tag{100}$$
$$= w_{00}(f_{00}(z), z)\, F'(z) + f_{00}'(z) \frac{\partial w_{00}(f_{00}(z), z)}{\partial x} F(z)$$
$$+ f_{00}'(z) W(f_{00}(z), z) .$$

The three boundary conditions from Eq. 84 ($\sigma_{ij} n_j = 0$) will be evaluated at $0(F_r^{-1})$. These three conditions are:

$$n_x \sigma_{\theta x} + n_\theta \sigma_{\theta\theta} + n_z \sigma_{\theta z} = 0 , \tag{101}$$

$$n_x \sigma_{xx} + n_\theta \sigma_{x\theta} + n_z \sigma_{xz} = 0 , \tag{102}$$

and

$$n_x \sigma_{zx} + n_\theta \sigma_{z\theta} + n_z \sigma_{zz} = 0 . \tag{103}$$

These three expressions are analogously compared to Eq. 94 at $0(F_r^{-1})$ to show the effects of gravity. They are, respectively:

$$F(z)\,\frac{\partial^2 v_{00}(f_{00}(z),\,z)}{\partial x^2}+\frac{\partial V(f_{00}(z),\,z)}{\partial x} \tag{104}$$

$$=f_{00}'(z)\;F'(z)\,\frac{\partial v_{00}(f_{00}(z),\,z)}{\partial x}$$

$$+\,f_{00}'(z)\;F(z)\,\frac{\partial^2 v_{00}(f_{00}(z),\,z)}{\partial x\partial z}$$

$$+\,f_{00}'(z)\;V(f_{00}(z),\,z)$$

$$+\,F'(z)\,\frac{\partial v_{00}(f_{00}(z),\,z)}{\partial z}\,,$$

$$-\,ReF(z)\,\frac{\partial P_{00}(f_{00}(z),\,z)}{\partial x}-ReP(f_{00}(z),\,z) \tag{105}$$

$$+\,2\,F(z)\,\frac{\partial^2 u_{00}(f_{00}(z),\,z)}{\partial x^2}+2\,\frac{\partial U(f_{00}(z),\,z)}{\partial x}=$$

$$+\,f_{00}'(z)\;F'(z)\,\frac{\partial u_{00}(f_{00}(z),\,z)}{\partial x}$$

$$+\,f_{00}'(z)\;F(z)\,\frac{\partial^2 u_{00}(f_{00}(z),\,z)}{\partial x\partial z}$$

$$+\,f_{00}'(z)\,\frac{\partial U(f_{00}(z),\,z)}{\partial z}$$

$$+\,f_{00}'(z)\;F(z)\,\frac{\partial^2 w_{00}(f_{00}(z),\,z)}{\partial x^2}$$

$$+\,f_{00}'(z)\;U(f_{00}(z),\,z)$$

$$+\,F'(z)\,\frac{\partial u_{00}(f_{00}(z),\,z)}{\partial z}$$

$$+\,F'(z)\,\frac{\partial w_{00}(f_{00}(z),\,z)}{\partial z}\,,$$

and

$$F'(z)\,\frac{\partial u_{00}(f_{00}(z),\,z)}{\partial x}+F(z)\,\frac{\partial^2 u_{00}(f_{00}(z),\,z)}{\partial x\partial z} \tag{106}$$

$$+\,\frac{\partial U(f_{00}(z),\,z)}{\partial z}$$

$$+ F(z) \frac{\partial^2 w_{00}(f_{00}(z), z)}{\partial x^2}$$

$$+ \frac{\partial W(f_{00}(z), z)}{\partial x} + f_{00}'(z)\, F(z) \frac{\partial P_{00}(f_{00}(z), z)}{\partial x}$$

$$+ f_{00}'(z)\, Re\, P_{00}(f_{00}(z), z)$$

$$- 2 f_{00}'(z)\, F'(z) \frac{\partial w_{00}(f_{00}(z), z)}{\partial x}$$

$$- 2 f_{00}'(z)\, F(z) \frac{\partial^2 w_{00}(f_{00}(z), z)}{\partial x \partial z}$$

$$- 2 f_{00}'(z) \frac{\partial W(f_{00}(z), z)}{\partial z}$$

$$+ F(z)\, Re\, P_{00}(f_{00}(z), z)$$

$$- 2F(z) \frac{\partial w_{00}(f_{00}(z), z)}{\partial z} = 0$$

DISCUSSION AND CONCLUSIONS

The base axisymmetric flow solutions of the current collector axisymmetric flow field are designated as u_{00}, v_{00}, and w_{00}, which are the nondimensional radial, azimuthal, and axial velocity components, respectively. Gravity perturbs the axisymmetric base flow solutions of the current collector flow field by making the flow nonaxisymmetric. Gravity accelerates or retards the base flow at various azimuthal positions. The nondimensional, nonaxisymmetric asymptotic expansions for the velocity components perturbed by gravity are determined to be of the form:

$$u = u_{00} + F_r^{-1} \sin\theta\, U(x,z) \ , \qquad (107)$$

$$v = v_{00} + F_r^{-1} \sin\theta\, V(x,z) \ ,$$

and

$$w = w_{00} + F_r^{-1} \sin\theta\, W(x,z) \ ,$$

where the gravity perturbation term is of order $0(F_r^{-1})$ and where F_r is the nondimensional Froude number (the ratio of the inertial force divided by the gravity force). The gravity term is also a function of $\sin\theta$, where $\theta = 0°$ is along the horizontal. The functions U, V, and W are only functions of x and z, and not θ.

Work is presently being performed by Audet *et al.*[6] to determine the base flow current collector solutions without gravity perturbations u_{00}, v_{00}, and w_{00} numerically from Eqs. 31 through 34 with appropriate boundary conditions including the surface tension effects (Eqs. 53 through 56). The first condition at the free surface requires that the velocities of the fluid vectors in the direction of the unit normal at the free surface be equal to each other and to zero. The second condition requires the forces which the fluids exert on each other to be opposite. In this work the stresses in the inert cover gas are assumed to be small enough to be neglected. The functions U, V, and W, the functions for the gravity perturbations, are governed by four coupled linear partial differential equations

(Eqs. 78 through 81), which are functions of the base solutions u_{00}, v_{00}, and w_{00} as variable coefficients. The gravity-perturbed boundary conditions are determined by Eq. 82 and the linear boundary conditions (Eqs. 100 and 104 through 106). This complex system of equations will be used in future work to calculate the effect of gravity on the base flow of the current collector in order to determine the effects of gravity on the linear stability analysis that determines the collector ejection speed. Woo *et al.*[1] have performed the linear stability analysis without gravity.

APPENDIX
REDERIVATION OF LINEAR STABILITY ANALYSIS WITHOUT GRAVITY EFFECTS

INTRODUCTION

This Appendix presents a clearer, more precise, and somewhat different derivation of the hydrodynamic instability threshold of a liquid metal current collector than that presented by Woo *et al.*[1] This derivation also confirms the theory developed previously for the hydrodynamic ejection threshold. At high rotor speeds, the liquid metal current collector is subject to flow instabilities leading to almost instantaneous ejection of the liquid metal from the current-carrying region and disrupting the operation of the device. The basic geometry considered in this study[1] is the classic tongue-and-groove illustrated schematically in the cross sectional view in Fig. 4, in which an inner disk is fitted into the circumferential groove of the outer disk. Generally, the grooved piece is the stator, and the disk is the rotor. In Fig. 4, the radial gap is of width d; the axial gap of length a between the disk and the grooved piece are exaggerated for clarity. The volume in the radial gap is filled with liquid metal, which is set into rotational motion by the drag from the rotation of the inner disk and which is maintained in the grooved volume by centrifugal force. Thus, an electrical current path is maintained between the disk (rotor) and the groove (stator). To minimize both electrical and viscous power losses, the volume of the liquid metal occupies a minimum of the axial gap. As a result, when the liquid metal is fully disturbed in the collector, there is a free surface in the axial gap between the liquid metal and a nonreactive cover gas. Generally, the radius to the liquid metal surface R is much greater than d and a. The cover gas[1] is also in rotational motion due to drag, and, as with the liquid metal, the azimuthal flow is a function of the position along the axis because the boundary conditions require the cover gas flow to match both the moving and stationary surfaces. The centrifugal forces cause secondary circulation flows in both the liquid metal and the cover gas. The meridional secondary flows form vortices superposed on the azimuthal primary flow. In general, the stability of the current collector flow field in response to perturbations is of concern. The stability of the liquid metal collector from hydrodynamic first principles[1] will be discussed in this Appendix. Magnetohydrodynamic effects on the current collector ejection will be considered in later work by the authors. Further information can be obtained from the efforts of Eriksson,[7] who pioneered earlier work on this problem. ***The variables used in the Appendix are defined as needed for the reader, and are not necessarily those used in the text.***

PRIMARY FLOW FIELDS

The dynamics of the current collector system in Fig. 4 are described by the dimensional Navier-Stokes equation and the continuity equation in cyclindrical coordinates. Thus,

$$\varrho\left(\frac{Du}{Dt}-\frac{v_\theta^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right), \tag{108}$$

$$\varrho\left(\frac{Dv}{Dt}+\frac{uv_\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\mu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial\theta}\right),$$

and

$$\varrho\frac{Dw}{Dt}=-\frac{\partial P}{\partial z}+\mu\nabla^2 w, \tag{110}$$

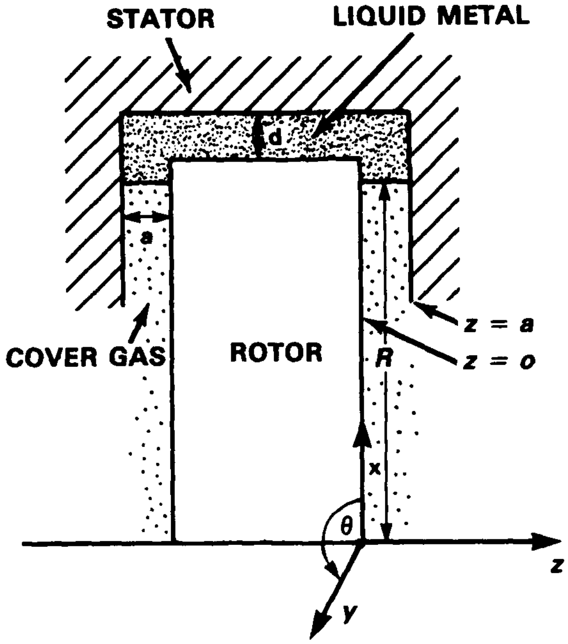

Fig. 4. Typical current collector configuration.[1]

where the substantial derivative D/Dt and the Laplacian ∇^2 are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}, \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{111}$$

and the continuity equation is

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0 . \tag{112}$$

For steady state, axisymmetric flow, derivatives with respect to time t and azimuthal angle are zero. The fluid has a viscosity μ and a density ϱ.

The dimensional variables are defined as:

(u,v_θ,w) = components of the velocity field in the radial, azimuthal, and axial directions.;

(r,θ,z) = components of cylindrical coordinates in radial, azimuthal, and axial directions.

P = scalar pressure component.

The conducting fluid flowing between the rotating disk and the stationary groove is contained absolutely except at the interface with the inert cover gas between the azimuthally rotating disk and stationary sidewall. Although the situation will become unstable for sufficiently large differencies between the azimuthal velocities in the liquid metal and cover gas, stable equilibrium will be established at lower azimuthal velocities. In the mathematical model, the radial gap between the stator and rotor is assumed to be zero. The gas is assumed to be incompressible and occupies the region $0 \leqslant r \leqslant R$. The liquid metal occupies the region $R \leqslant r \leqslant R + d$. The gas and liquid base velocities are assumed to be similar to two different rigid body rotations with no axial variations. V_L and V_G

represent the interface liquid, and gas primary flow velocities at $r = R$, respectively, which are constants. The primary solutions for the velocity and pressure for the liquid $i = L$ and gas $i = G$, respectively, are:

$$V_{\theta L} = V_L \left(\frac{r}{R}\right), \; P_L = \frac{\varrho_L V_L^2}{2}\left(\frac{r^2}{R^2} - 1\right) \tag{113}$$

$$V_{\theta G} = V_G \left(\frac{r}{R}\right), \; P_G = \frac{\varrho_G V_G^2}{2}\left(\frac{r^2}{R^2} - 1\right) + T/R \tag{114}$$

where T is the surface tension between liquid and gas.

The primary pressure in the liquid P_L and gas P_G is obtained from the radial component of the Navier-Stokes equation (Eq. 108) by equating the centrifugal force term $\frac{\varrho v_\theta^2}{r}$ to the radial gradient of the pressure, neglecting the other terms, and integrating between 0 and r. After substituting $V_{\theta i} = V_i \left(\frac{r}{R}\right)$, integration provides

$$\frac{1}{\varrho} P(r) = \frac{V_i r^2}{2R^2} + \text{constant.} \tag{115}$$

Arbitrarily setting the pressure of the liquid P_L to 0 at interface $r = R$ determines the constants in both equations for the liquid and gas and results in Eqs. 113 and 114.

PERTURBATION OF VELOCITY AND PRESSURE

The perturbation expansions are restricted to have no axial variation, $\frac{\partial}{\partial z} = 0$, and no axial velocity, $v_z = 0$. The expansions for the liquid velocity $\underset{\sim}{v}_L$, liquid pressure P_L, gas pressure $\underset{\sim}{v}_G$, and gas pressure P_G are the two first orders in ε.
Thus,

$$\underset{\sim}{v}_L = \varepsilon\, \hat{v}_{rL}(r,\theta,t)\, \underset{\sim}{\hat{r}} + \left[V_L \left(\frac{r}{R}\right) + \varepsilon\, \hat{v}_{\theta L}(r,\theta,t) \right] \underset{\sim}{\hat{\theta}} \,, \tag{116}$$

$$P_L = \frac{\varrho_L V_L^2}{2}\left(\frac{r^2}{R^2} - 1\right) + \varepsilon\, \hat{P}_L(r,\theta,t) \,, \tag{117}$$

$$\underset{\sim}{v}_G = \varepsilon\, \hat{v}_{rG}(r,\theta,t)\, \underset{\sim}{\hat{r}} + \left[V_G \left(\frac{r}{R}\right) + \varepsilon\, \hat{v}_{\theta G}(r,\theta,t) \right] \underset{\sim}{\hat{\theta}} \,, \tag{118}$$

and

$$P_G = \frac{\varrho_G V_G^2}{2}\left(\frac{r^2}{r^2} - 1\right) + \frac{T}{R} + \varepsilon\, \hat{P}_G(r,\theta,t) \,, \tag{119}$$

where ε is a small parameter much less than 1, i.e., $\varepsilon << 1$.

Since gravity is neglected, the problem has axisymmetric symmetry. The centrifugal force in the system due to the rotational motion of the fluid is the dominant equilibrium force. Viscous terms are neglected in this analysis because this is an inviscid analysis. Only $0(\varepsilon)$ terms will be kept in this analysis, and nonlinear $0(\varepsilon^2)$ terms and those of higher order will be dropped because this is a linear analysis for small perturbations for $\varepsilon << 1$. The hydrodynamic stability of the collector interface at $r = R$ between the liquid and

cover gas in relative motion was studied by the perturbed Navier-Stokes equation and the perturbed continuity equation to first order in ε for both fluids.
Thus,

$$\varrho_i \left[\frac{\partial \hat{v}_{ri}}{\partial t} + \frac{V_i}{R} \frac{\partial \hat{v}_{ri}}{\partial \theta} - \frac{2V_i}{R} \hat{v}_{\theta i} \right] = \frac{\partial \hat{P}_i}{\partial r} , \tag{120}$$

$$\varrho_i \left[\frac{\partial \hat{v}_{\theta i}}{\partial t} + \frac{2V_i}{R} \hat{v}_{ri} + \frac{V_i}{R} \frac{\partial \hat{v}_{\theta i}}{\partial \theta} \right] = - \frac{1}{r} \frac{\partial \hat{P}_i}{\partial \theta} , \tag{121}$$

and

$$\frac{\partial \hat{v}_{ri}}{\partial r} + \frac{\hat{v}_{ri}}{r} + \frac{1}{r} \frac{\partial \hat{v}_{\theta i}}{\partial \theta} = 0 . \tag{122}$$

There is no axial component to the Navier-Stokes equation since there is no axial velocity perturbation. These equations were derived by substituting Eqs. 116 through 119 in Eqs. 108 through 112 and retaining the terms to order ε.

BOUNDARY CONDITIONS AT NONLINEAR FREE SURFACE

The free surface position for the primary flow at $r = R$ is perturbed to

$$r = R + \varepsilon f(\theta,t) . \tag{123}$$

Before the boundary conditions for the linearized case are considered, the fully nonlinearlized boundary conditions will be derived. The nonlinear free surface condition can be expressed as

$$r = F(\theta,t) , \tag{124}$$

where the liquid is in the region

$$r \geqslant F(\theta,t) . \tag{125}$$

The velocity of the liquid v_L and the pressure of the liquid P_L are, respectively:

$$\underset{\sim}{v}_L = v_{rL}(r,\theta,t)\, \hat{\underset{\sim}{r}} + v_{\theta L}(r,\theta,t)\, \hat{\underset{\sim}{\theta}} , \tag{126}$$

and

$$P_L = P_L(r,\theta,t) , \tag{127}$$

The velocity of the gas v_G and the pressure of the gas P_G are, respectively:

$$\hat{v}_G = v_{rG}(r,\theta,t)\, \hat{\underset{\sim}{r}} + v_{\theta G}(r,\theta,t)\, \hat{\underset{\sim}{\theta}} , \tag{128}$$

and

$$P_G = P_G(r,\theta,t) . \tag{129}$$

The stress tension σ_{ij} in an incompressible inviscid fluid takes the simple form for the liquid L and gas G, respectively,

$$\sigma_{ijL} = - P_L \, \delta_{ij} , \tag{130}$$

and

$$\sigma_{ijG} = - P_G \, \delta_{ij} . \tag{131}$$

Since there are no shear stresses, there are no restrictions on tangential velocities at the free surface. Since the fluids are considered to be inviscid, the liquid metal and cover gas can slide over each other at the free surface. The inviscid nonlinear boundary conditions at $r = F(\theta,t)$ are:

$$\underset{\sim}{v}_L \cdot \hat{\underset{\sim}{n}} = \underset{\sim}{v}_G \cdot \hat{\underset{\sim}{n}} = v_s \cdot \hat{\underset{\sim}{n}} \ , \tag{132}$$

and

$$P_G = P_L + \frac{T}{R_c} \text{ (normal stress condition)} \ , \tag{133}$$

where

$\hat{\underset{\sim}{n}}$ = local unit normal to free surface,

$\underset{\sim}{v}_s$ = velocity of free surface position, i.e.,

$$\underset{\sim}{v}_s = \frac{\partial F(\theta,t)}{\partial t} \hat{\underset{\sim}{r}} \ , \tag{134}$$

and

R_c = local radius of curvature of free surface defined as positive when radius of curvature is on the gas side of the surface.

The local unit normal to the free surface can now be derived, and the function can be defined as

$$\Phi(r,\theta,t) = r - F(\theta,t) \ , \tag{135}$$

so that $\Phi = 0$ at the free surface. $\underset{\sim}{\nabla}\Phi$ is perpendicular to the free surface at $r = F(\theta,t)$. Therefore, the gradient of Φ is

$$\underset{\sim}{\nabla}\Phi = \hat{\underset{\sim}{r}} \quad - \frac{1}{r}\frac{\partial F(\theta,t)}{\partial \theta} \hat{\underset{\sim}{\theta}} \ , \tag{136}$$

where

$$\underset{\sim}{\nabla} = \hat{\underset{\sim}{r}} \frac{\partial}{\partial r} + \hat{\underset{\sim}{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} \ . \tag{137}$$

The gradient of F at $r = F(\theta,t)$ along the free surface is

$$\underset{\sim}{\nabla}\Phi \Big|_{r = F(\theta,t)} = \hat{\underset{\sim}{r}} \quad - \frac{1}{F(\theta,t)} \frac{\partial F(\theta,t)}{\partial \theta} \hat{\underset{\sim}{\theta}} \ . \tag{138}$$

Thus, the local unit normal along the free surface $\hat{\underset{\sim}{n}}$ is equal to

$$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{\nabla}\Phi \Big|_{r=F(\theta,t)}}{\left| \underset{\sim}{\nabla}\Phi \right|_{r=F(\theta,t)}} = \frac{\hat{\underset{\sim}{r}} - \dfrac{1}{F(\theta,t)} \dfrac{\partial F(\theta,t)}{\partial \theta} \hat{\underset{\sim}{\theta}}}{\left[1 + \dfrac{1}{F^2} \left(\dfrac{\partial F}{\partial \theta} \right)^2 \right]^{1/2}} \tag{139}$$

$$= \left[F^2 + \left(\frac{\partial F}{\partial \theta} \right)^2 \right]^{-1/2} \left[F \hat{\underset{\sim}{r}} - \frac{\partial F}{\partial \theta} \hat{\underset{\sim}{\theta}} \right] \ . \tag{140}$$

Using Eq. 132, the nonlinear velocity boundary condition is found to be

$$v_{ri} = \frac{1}{F}\frac{\partial F}{\partial \theta} v_{\theta i} + \frac{\partial F}{\partial t} \qquad \text{at } r = F(\theta,t) \ . \tag{141}$$

The radius of curvature at each point along the interface $r = F(\theta,t)$ must now be derived. A line $r = F(\theta,t)$ in the $z = 0$ plane is now considered (Fig. 5). Let s equal the distance along the curve from some point, $s = 0$. $\hat{\underset{\sim}{n}}(s)$ is the local unit normal at each point on the curve. The distance ds equals $R_c d\alpha$, where α is the angle subtended by ds .

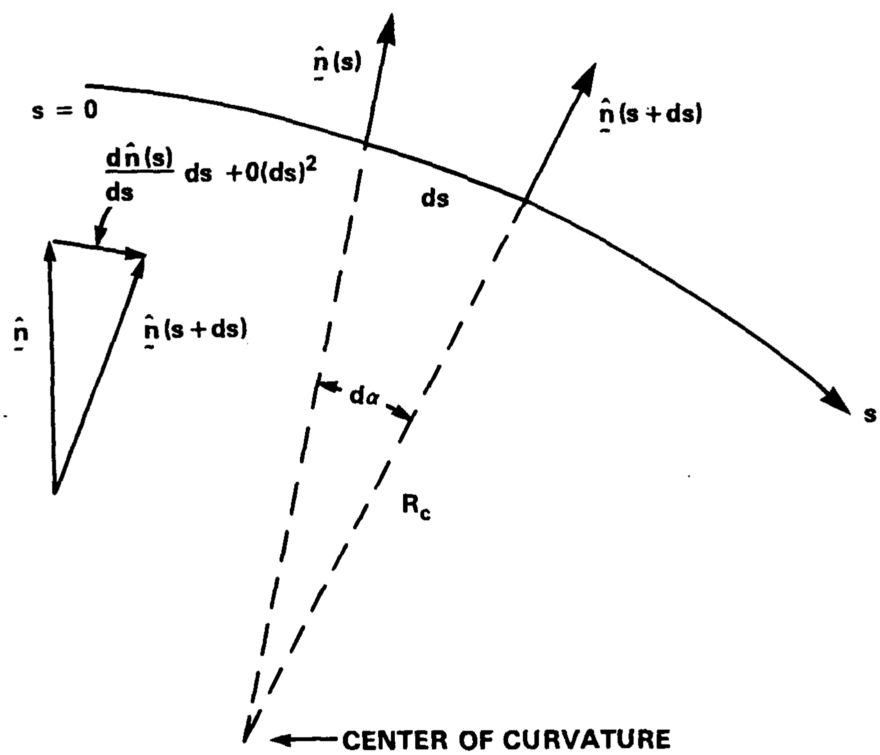

Fig. 5. Radius of curvature at each point.

The unit normal at $\hat{n}(s + ds)$ can be expressed in terms of $\hat{n}(s)$,
Thus,

$$\hat{n}(s+ds) = \hat{n}(s) + ds \frac{d\hat{n}(s)}{ds} + 0\,(ds)^2 + \ldots \tag{142}$$

in a Taylor series expansion. By vector subtraction

$$\hat{n}(s+ds) - \hat{n}(s) = \frac{d\hat{n}(s)}{ds}\,ds + 0\,(ds)^2 + \ldots . \tag{143}$$

Since $|\ \hat{n}(s)\ | = 1$, $\hat{n}(s)$ can change direction, i.e., rotation, but not length (Fig. 5). Therefore, $\frac{d\hat{n}(s)}{ds}$ is perpendicular to $\hat{n}(s)$. From similar triangles the identity

$$\left|\frac{d\hat{n}(s)}{ds}\right| ds = |\ \hat{n}(s)\ |\ d\alpha \tag{144}$$

can be obtained, from which the radius of curvature $R_c(s)$ can be derived

$$\frac{1}{R_c(s)} = \left|\frac{d\hat{n}(s)}{ds}\right| , \tag{145}$$

since

$$\left|\frac{d\hat{n}(s)}{ds}\right| ds = d\alpha = \frac{ds}{R_c} . \tag{146}$$

The curve in the $z = 0$ plane is $r = F\ (\theta,t)$. The two-dimensional metric in cylindrical coordinates (r,θ) is

$$ds = [(dr)^2 + r^2(d\theta)^2]^{1/2} \text{ at } r = F(\theta,t) . \tag{147}$$

Since $dr = \frac{\partial F}{\partial \theta}\,d\theta$ at a particular time, ds can be expressed again as

$$ds = \left[r^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{1/2} d\theta . \tag{148}$$

The derivative $\frac{d\theta}{ds}$ is, simply,

$$\frac{d\theta}{ds} = \left[\mathrm{F}^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} , \tag{149}$$

and the derivative $\frac{d\hat{n}}{ds}$ is

$$\frac{d\hat{n}}{ds} = \frac{d\hat{n}}{d\theta}\frac{d\theta}{ds} = \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} \frac{d\hat{n}}{d\theta} . \tag{150}$$

Actually, the $\hat{n}$ derived from Eqs. 139 and 140 is a function of θ and t, but the curvature at each point on the curve for one particular time is being addressed, so t is fixed here. Thus,

$$\hat{\underset{\sim}{n}} = \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} \left[F\hat{\underset{\sim}{r}} - \left(\frac{\partial F}{\partial \theta}\right)\hat{\underset{\sim}{\theta}}\right] . \tag{151}$$

The derivative of $\hat{\underset{\sim}{n}}$ with respect to θ is

$$\frac{d\hat{\underset{\sim}{n}}}{d\theta} = \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} \left[\frac{\partial F}{\partial \theta}\hat{\underset{\sim}{r}} + F\hat{\underset{\sim}{\theta}}\right] \tag{152}$$

$$\times \left\{1 + \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1}\left[\left(\frac{\partial F}{\partial \theta}\right)^2 - F\frac{\partial^2 F}{\partial \theta^2}\right]\right\},$$

where *it* is remembered so that

$$\frac{d\hat{\underset{\sim}{r}}}{d\theta} = \hat{\underset{\sim}{\theta}} , \text{ and } \frac{d\hat{\underset{\sim}{\theta}}}{d\theta} = - \hat{\underset{\sim}{r}} . \tag{153}$$

From Eqs. 150 and 152, the radius of curvature of the curve at $r = F(\theta,t)$ at a given instant is

$$\frac{1}{R_c} = \left|\frac{d\hat{\underset{\sim}{n}}}{ds}\right| = \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} \left|\frac{d\hat{\underset{\sim}{n}}}{d\theta}\right| \tag{154}$$

$$= \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2}\left\{1 + \left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1}\right.$$

$$\left.\times\left[\left(\frac{\partial F}{\partial \theta}\right)^2 - F\frac{\partial^2 F}{\partial \theta^2}\right]\right\}.$$

Now, the only question is whether the correct sign for this convention is obtained. If $F(\theta,t) = R$, Eq. 154 provides $\frac{1}{R_c} = \frac{1}{R}$, which is correct. Therefore, the boundary condition for the liquid and gas pressure at the interface is

$$P_G = P_L + T/R_c \qquad \text{at } r = F(\theta,t) . \tag{155}$$

These are the boundary conditions for a nonlinear analysis.

LINEAR BOUNDARY CONDITIONS AT A FREE SURFACE

The nonlinear free surface was represented as $r = F(\theta,t)$. A linear representation of the free surface would be $r = R + \varepsilon f(\theta,t)$. The nonlinear free surface velocity boundary condition was derived previously as

$$v_{ri} = \frac{1}{F}\frac{\partial F}{\partial \theta}v_{\theta i} + \frac{\partial F}{\partial t} \qquad \text{at } r = F(\theta,t), \tag{156}$$

where $i = L$ or G and $v_i = \varepsilon \underset{\sim}{v}_{ri}(r,\theta,t)\,\hat{\underset{\sim}{r}} + \left[V_i\left(\frac{r}{R}\right) + \varepsilon\underset{\sim}{v}_{\theta i}(r,\theta,t)\right]\hat{\underset{\sim}{\theta}}.$ (157)

The boundary condition to order $0(\varepsilon)$can be expressed as

$$\varepsilon\hat{v}_{ri}\,[R + \varepsilon f(\theta,t),\theta,t] = [R + \varepsilon f(\theta,t)]^{-1}\varepsilon\frac{\partial f(\theta,t)}{\partial \theta} \tag{158}$$

$$\times\left\{V_i\left[1 + \frac{\varepsilon f(\theta,t)}{R}\right] + \varepsilon\hat{v}_{\theta L}\,[R + \varepsilon f(\theta,t)]\right\}$$

$$+ \varepsilon\frac{\partial f(\theta,t)}{\partial t}$$

Introducing the well known Taylor series:

$$\hat{v}_{ri}\,[R + \varepsilon\, f(\theta,t),\theta,t] = \hat{v}_{ri}\,(R,\theta,t) \tag{159}$$

$$+ \varepsilon\, f(\theta,t)\,\frac{\partial \hat{v}_{ri}}{\partial r}(R,\theta,t) + \tfrac{1}{2}\,\varepsilon^2 f^2\,\frac{\partial^2 \hat{v}_{ri}}{\partial r^2}(R,\theta,t) + \ldots\,,$$

and

$$\hat{v}_{\theta i}\,[R + \varepsilon f(\theta,t),\theta,t] = \hat{v}_{\theta i}\,(R,\theta,t) \tag{160}$$

$$+ \varepsilon\, f(\theta,t)\,\frac{\partial \hat{v}_{\theta i}}{\partial r}(R,\theta,t) + \tfrac{1}{2}\,\varepsilon^2 f^2(\theta,t)\,\frac{\partial^2 \hat{v}_{\theta i}}{\partial r^2}(R,\theta,t) + \ldots\,.$$

Substituting these expressions in Eq. 158 and neglecting terms of $O\,(\varepsilon^2)$:

$$\hat{v}_{ri}\,(R,\theta,t) = \frac{V_i}{R}\,\frac{\partial f(\theta,t)}{\partial \theta} + \frac{\partial f(\theta,t)}{\partial t}\,, \tag{161}$$

for $i = L$ or G.

It is of note that the boundary condition is applied to v_{ri} at $r = R$ because the Taylor series was about the point $r = R$.

The expression for the radius of curvature to order ε is obtained from Eq. 154 and is equal to

$$\frac{1}{R_c} = \frac{1}{R} - \frac{\varepsilon}{R^2}\left[\frac{\partial^2 f(\theta,t)}{\partial \theta^2} + f(\theta,t)\right] + 0\,(\varepsilon^2)\,, \tag{162}$$

where the following expressions were used:

$$\left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right] = R^2 + 2R\varepsilon f(\theta,t) + 0(\varepsilon^2)\,, \tag{163}$$

$$\left[F^2 + \left(\frac{\partial F}{\partial \theta}\right)^2\right]^{-1/2} = \frac{1}{R} - \frac{\varepsilon}{R^2} f(\theta,t) + 0\,(\varepsilon^2)\,, \tag{164}$$

and

$$\left(\frac{\partial F}{\partial \theta}\right)^2 - F\,\frac{\partial^2 F}{\partial \theta^2} = -\,\varepsilon R\,\frac{\partial^2 f(\theta,t)}{\partial \theta^2} + 0\,(\varepsilon^2)\,. \tag{165}$$

The linearized normal stress condition at the free surface is obtained from the nonlinear surface condition

$$P_G = P_L + \frac{T}{R_c}\,, \tag{166}$$

where

$$P_L = \frac{\varrho_L V_L^2}{2}\left(\frac{r^2}{R^2} - 1\right), \tag{167}$$

and

$$P_G = \frac{\varrho_G V_G^2}{2}\left(\frac{r^2}{R^2} - 1\right) + \frac{T}{R}\,. \tag{168}$$

After simple substitution and the use of the linearized free surface condition $r = R + \varepsilon f(\theta,t)$, the expression to order ε for the reciprocal of the curvature, (Eq. 162) and the Taylor series expansion of the linearized normal stress condition at the free surface is

$$\frac{\varrho_G V_G^{\,2}}{R} f(\theta,t) + \hat{P}_G\,(R,\theta,t) = \frac{\varrho_L V_L^{\,2}}{R} f(\theta,t) \tag{169}$$

$$+ \hat{P}_L(R,\theta,t) - \frac{T}{R^2}\left[\frac{\partial^{\,2} f(\theta,t)}{\partial\theta^2} + f(\theta,t)\right] .$$

PERIODIC SOLUTION IN THE AZIMUTHAL ANGLE AND TIME

Since θ is the azimuthal angular coordinate, the following substitution shall be made:

$$\hat{P}_i(r,\theta,t) = P_i(r)\ exp\ (in\pi\theta + i\omega t)\ . \tag{170}$$

All integer values shall be considered and the value of n corresponding to the first unstable mode shall be found. The n for the first unstable mode is very large, thus, n can be treated as a continuous variable rather than a discrete variable with only integer values. Therefore, the discrete variable $n\pi$ shall be replaced with the continuous variable kR. Explained differently, the first unstable mode has a wavelength $2\pi/k$ for the first unstable mode k. The circumference of the interface is $2\pi R$. If the wavelength is much smaller than the circumference

$$\frac{2\pi}{k} << 2\pi R \rightarrow kR >> 1\ , \tag{171}$$

then the disturbance is unaffected by the finite circumference and is the same as a disturbance on an infinite length surface.

The simple transformation is introduced:

$$r = R + x \text{ and } \frac{\partial}{\partial r} = \frac{\partial}{\partial x}\ , \tag{172}$$

where no assumptions on the size of x have been determined yet in the work. The perturbed quantities are assumed to be of the exponential form

$$\hat{v}_{ri}(r,\theta,t) = U_{ri}(x)\ exp\ [i(kR\theta + \omega t)]\ , \tag{173}$$

$$\hat{v}_{\theta i}(r,\theta,t) = U_{\theta i}(x)\ exp\ [i(kR\theta + \omega t)]\ , \tag{174}$$

$$\hat{P}_i(r,\theta,t) = P_i(x)\ exp\ [i(kR\theta + \omega t)]\ , \tag{175}$$

and

$$f(\theta,t) = A\ exp\ [i(kR\theta + \omega t)]\ , \tag{176}$$

where $i = L$ or G (liquid or gas).

Substituting Eqs. 173 through 176 the perturbed Navier-Stokes equations to order ε results in the following set of ordinary coupled differential equations:

$$\varrho_i\left[i(\omega + kV_i)\ U_{ri}(x) - \frac{2V_i}{R}\ U_{\theta L}(x)\right] = -\frac{\partial P_i}{\partial x}\ , \tag{177}$$

$$\varrho_i\left[i(\omega + kV_i)\ U_{\theta i}(x) + \frac{2V_i}{R}\ U_{ri}(x)\right] = -\frac{ikR}{(R+x)} P_i\ , \tag{178}$$

and

$$\frac{dU_{ri}(x)}{dx} + \frac{U_{ri}(x)}{(R+x)} + \frac{ikR}{(R+x)}\ U_{\theta i}(x) = 0\ . \tag{179}$$

In Eq. 177 the term $ikV_iU_{ri} >> \frac{2V_i}{R} U_{\theta i}$, and in Eq. 178 the term $ikV_iU_{\theta i} >> \frac{2V_i}{R} U_{ri}$. In each case, the smaller term is neglected. These inequalities were determined by noting $kR>>1$.

The radial penetration of the disturbance from the free surface shall be shown to be comparable to the azimuthal wavelength, i.e., $\frac{2\pi}{k}$, $\frac{d}{dx}$ is comparable to k. Noting that $R >>x$, the term U_{ri}/R is neglected in Eq. 179. Equations 177 through 179 are now simplified to the following set of coupled differential equations:

$$i\varrho_i(\omega + k\,V_i)\,U_{ri} = -\frac{dP_i}{dx} \tag{180}$$

$$i\varrho_i(\omega + k\,V_i)\,U_{\theta i} = -ikP_i\ , \tag{181}$$

and

$$\frac{dU_{ri}}{dx} + ik\,U_{\theta i} = 0\ . \tag{182}$$

From these three equations, the following simple relationships are obtained:

$$U_{\theta i} = \frac{i}{k}\frac{dU_{ri}}{dx}\ . \tag{183}$$

$$P_i = -\frac{i\varrho_i}{k^2}(\omega + k\,V_i)\frac{dU_{ri}}{dx}\ , \tag{184}$$

and

$$\frac{d^2U_{ri}}{dx^2} = k^2U_{ri}\ . \tag{185}$$

The general solution to the ordinary differential equation (Eq. 185) is

$$U_{ri} = B_1 exp\,(kx) + B_2\,exp\,(-kx)\ , \tag{186}$$

where it is assumed that $k > 0$. For the gas ($i = G$), the following relationships hold: $-R \leqslant x \leqslant 0$ and $\frac{d}{dx} = k \rightarrow \Delta x = \frac{1}{k}$, where Δx equals the radial extent of the disturbance from the free surface. In the gas $R/(\Delta x) = kR >>1$ and x is located effectively in the region $-\infty < x \leqslant 0$ and $U_{rG} \rightarrow 0$ as $x = -\infty$, implying $B_z = 0$. Therefore, using Eqs. 173 through 176 and Eqs. 183 and 184, the following relationships can be obtained for the gas phase:

$$U_{rG}\,(x) = B_1\,exp\,(kx)\ , \tag{187}$$

$$U_{\theta G}\,(x) = iB_1\,exp\,(kx)\ , \tag{188}$$

and

$$P_G\,(x) = -\frac{i\varrho_G}{k}B_1\,(\omega + kV_G)\,exp\,(kx)\ . \tag{189}$$

For the liquid ($i = L$), the following relationships hold $0 \leqslant x \leqslant d$, $\frac{d}{\Delta x} \sim kd$. If the radial distance $d >> \Delta x$ azimuthal wavelength $\frac{2\pi}{k}$ is assumed, then $kd >> 1$. Therefore, $d >> \Delta x$ and x

is effectively in the region $0 \leq x < \infty$ and $U_{rL} \rightarrow 0$ as $x \rightarrow \infty$, impling $B_1 = 0$. Therefore, from Eqs. 183 through 185, the following equations are obtained from the liquid phase:

$$U_{rL}(x) = B_2 \, exp\,(-kx) \; , \tag{190}$$

$$U_{\theta L}(x) = -\, iB_2 \, exp\,(-kx) \; , \tag{191}$$

and

$$P_r(x) = \frac{i\varrho_L}{k} (\omega + kV_L) \, B_2 \, exp\,(-kx) \; . \tag{192}$$

The solutions have been obtained, except for a constant in $U_{ri}(x)$, $U_{\theta i}(x)$, and $P_i(x)$, where $i = L$ or G for the gas and liquid phases in the current collector system.

Using the linearized velocity condition at the free surface for the liquid L and the gas G and after appropriate substitution, the following boundary conditions are obtained:

$$U_{rG}(0) = B_1 = i\,(\omega + kV_G)\,A \; , \tag{193}$$

$$U_{rL}(0) = B_2 = i\,(\omega + kV_L)\,A \; , \tag{194}$$

where

$$\frac{\partial^2 f(\theta,t)}{\partial \theta^2} = -\, k^2 R^2 A exp\,[i\,(kR\theta + \omega\, t)] \; , \tag{195}$$

and $f(\theta,t)$ in the surface term is neglected.

The linearized pressure condition at the free surface (Eq 169) has the following form after appropriate substitution:

$$\frac{\varrho_G}{k}(\omega + k\,V_G)^2 + \frac{\varrho_L}{k}(\omega + kV_L)^2 = \frac{\varrho_L V_L{}^2}{R} - \frac{\varrho_G V_G{}^2}{R} + k^2 T \; , \tag{196}$$

where

$$\omega = -\frac{k\,(\varrho_G V_G + \varrho_L V_L)}{(\varrho_L + \varrho_G)} \pm \left[\frac{k}{\varrho_G + \varrho_L} \left(\frac{\varrho_L V_L{}^2}{R} - \frac{\varrho_G V_G{}^2}{R} + k^2 T \right) - \frac{k^2 \varrho_G \varrho_L}{(\varrho_G + \varrho_L)^2} (V_G - V_L)^2 \right]^{1/2} . \tag{197}$$

Equation 197 was obtained from Eq. 196 by arranging it in quadratic form. The first term in Eq. 197 is real, and represents the normal mode of the system in the absence of driving forces. The quantities under the square root sign determine the stability of the system. The first term under the square root sign represents the difference in the centrifugal force term across the interface and surface tension between the liquid metal and cover gas. Both terms are stabilizing for the case of a liquid metal supported by a less dense cover gas. The second term under the square root is always destabilizing and is the source of the Kelvin-Helmholtz instability and the discontinuity in primary flow across the interface.

In Eq. 197, if the quantity Q under the square root $[Q]^{1/2}$ is greater than 0, $Q > 0$, the angular frequency ω is real and neutral stability is maintained in the system. However, if $Q < 0$, then $\omega = a \pm ib$ and the fluid flow system becomes unstable. In general, because of the dependence of quantities on the wave propagation constant k in Eq. 197, instabilities occur in the region of the k spectrum when the sum of the quantities under the

square root sign is negative, i.e., $Q < 0$. Thus, the stability threshold is obtained from

$$\left(\frac{\varrho_L V_L^2}{R} - \frac{\varrho_G V_G^2}{R} + k^2 T\right) > \frac{k\varrho_G \varrho_L}{(\varrho_G + \varrho_L)} (V_G - V_L)^2 , \tag{198}$$

which reduces to

$$(V_G - V_L)^2 < \frac{\varrho_L}{\varrho_G}\left(\frac{V_L^2}{kR} + \frac{kT}{\varrho_L}\right) , \tag{199}$$

when $\frac{\varrho_G V_G^2}{R}$ is neglected with respect to $\frac{\varrho_L V_L^2}{R}$ and set $\varrho_G + \varrho_L \approx \varrho_L$.

The wave number for the first unstable mode for the system k is derived by finding the minimum value of the right hand side of Eq. 199:

$$k = \left(\frac{\varrho_L V_L^2}{TR}\right)$$

For this k, the stability threshold equation (Eq. 199) is expressed as

$$(V_G - V_L)^2 < \frac{2(\varrho_L T)^{1/2} V_L}{\varrho_G R^{1/2}} . \tag{200}$$

If the azimuthal velocity profiles similar to those in Fig. 6 are assumed for the primary flow, the following approximations can be made where V_{ow} is the rotor velocity at the interface:

$$(V_G - V_L)^2 = \alpha^2 V_{ow}^2 , \tag{201}$$

and

$$V_L = \beta V_{ow} , \tag{202}$$

where $\alpha^2 = \beta$.

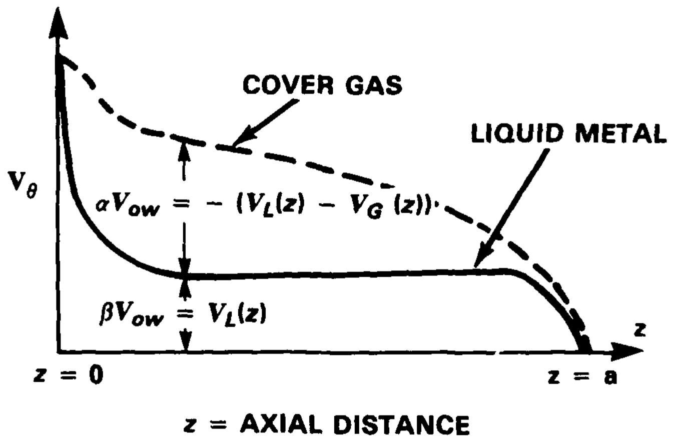

Fig. 6. Assumed typical azimuthal flow profile $V_o(z)$ at interface ($r = R$) based on similarity solution in base flow and conservation of angular momentum at the interface.[1]

Using these approximations in Eq. 200, the critical speed for ejection is obtained as

$$U_{\text{CRITICAL}} = (\Omega R)_{\text{CRITICAL}} = \frac{2(\varrho_L T)^{1/2}}{\varrho_G R^{1/2}} , \tag{203}$$

and

$$\Omega_{\text{CRITICAL}} = \frac{2(\varrho_L T)^{1/2}}{\varrho_G R^{3/2}} . \tag{204}$$

U_{CRITICAL} is the rotor speed at the interface in meters per second, and Ω_{CRITICAL} is the critical angular velocity at which hydrodynamic instability takes place. These results agree exactly with those obtained by Woo *et al.*,[1] by a somewhat different derivation.

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